ON THE LOCUS OF AN ANALYTIC EQUATION IN THE REAL PLANE*

BY A. B. BROWN

I have been unable to find in the literature a statement or proof of the following theorem.

THEOREM. *Let $f(x, y)$ be analytic at the real point (x_0, y_0), with $f(x_0, y_0)=0$ and $f(x, y)$ irreducible at (x_0, y_0).*† *Then the locus of the equation $f(x, y)=0$ in the real xy plane near (x_0, y_0) consists of one of the following three*: (1) *the point (x_0, y_0)*; (2) *a single smooth curve through (x_0, y_0)*; (3) *a cusp with vertex at (x_0, y_0).*

More detailed descriptions of (2) and (3) are contained in the proof which follows.

By a change of coordinates we may suppose $x_0=y_0=0$. According to the Weierstrass preparation theorem for the case of one independent variable, since f is irreducible at (0, 0), either $f(x, y)\equiv x\Omega(x, y)$, with $\Omega(0, 0)\neq 0$ and Ω analytic at (0, 0) or

$$(1)\qquad f(x, y) \equiv [y^m + A_1(x)y^{m-1} + \cdots + A_m(x)]\Omega(x, y),$$

with Ω as above, $m>0$, $A_j(x)$ analytic at $x=0$, and $A_j(0)=0$, $(j=1, \cdots, m)$. Since in the first case the real locus $f(x, y)=0$ is merely a straight line, it is sufficient to consider the case that (1) holds.

Since f is irreducible at (0, 0), the same is true of the algebroid function in (1), and hence its m-leaved Riemann surface is connected near (0, 0) and we can uniformize locally as follows:

$$(2)\qquad x = t^m,$$

$$(3)\qquad y = \psi(t) = a_1t + a_2t^2 + \cdots,$$

with ψ analytic at $t=0$, and a neighborhood of the origin in the

* Presented to the Society, February 23, 1935.

† That is, not the product, near (x_0, y_0) in the 4-space of the complex variables, of two functions each analytic and zero at (x_0, y_0). For theorems which we use involving functions of complex variables, see W. F. Osgood, *Lehrbuch der Funktionentheorie*, vol. 1, Chapter 8, §14, and vol. 2, part 1, Chapter 2, §§2, 4, 7, 9, 10, 11.