ON THE MATRIC EQUATIONS $P(X)=A$ AND $P(A, X)=0$

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1. *Introduction.* The matric equation

$$P(X) = A, \tag{1}$$

where $P(\lambda)$ is a polynomial with scalar coefficients and A is a given square matrix of order n, has received a good deal of attention within the past few years. The problem is to find square matrices X satisfying (1). This equation may possess solutions X which are expressible as polynomials in A. On the other hand, there may exist solutions, but none expressible as a polynomial in A; and finally, there are equations of the type (1) which possess no solution at all.*

In 1928 Roth† found necessary and sufficient conditions that there may exist solutions of (1) expressible as polynomials in A, and he found the number of such solutions, in case any exists. He employed the theory of elementary divisors. In this paper Roth gave a bibliography which was quite complete up to that time. In 1931 Franklin attacked the problem through the canonical form, and found not only all solutions X that are expressible as polynomials in A, but also solutions that are not so expressible. Rutherford‡ also employed the canonical form. Still more recently Ingraham§ discussed the problem using the theory of elementary divisors.

Let $F_j(A)$, $(j=0, \cdots, m)$ be known polynomials in A with scalar coefficients, and consider the more general equation

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* Franklin, *Algebraic matric equations*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 10 (1932), pp. 289–314.

† Roth, *A solution of the matric equation* $P(X)=A$, Transactions of this Society, vol. 30 (1928), pp. 579–596.

‡ Rutherford, *On the canonical form of a rational integral function of a matrix*, Proceedings Edinburgh Mathematical Society, (2), vol. 3 (1932), pp. 135–143.

§ Ingraham, *On the rational solutions of the matrix equation* $P(X)=A$, Journal of Mathematics and Physics, vol. 13 (1934), pp. 46–50.