SOME BESSEL FUNCTION EXPANSIONS

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This is a formal derivation of some asymptotic Bessel function expansions. An interesting feature of the derivation is that it makes use of Heaviside's *generalized exponential series.** The chief results arrived at, the Hankel function expansions (4) and (6), are generalizations of formulas (5) and (8) on page 141 in Watson's *Theory of Bessel Functions* in the same sense that Heaviside's generalized exponential series is a generalization of the ordinary exponential series.

In the course of some work on dipole radiation the author stumbled onto a particular case of the following relation

$$(1)\qquad H_0^{(1)}\left(z(1-a)^{1/2}\right) = \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu}^{(1)}(z).$$

In the particular case where ν is $1/2$, a is in the fourth quadrant and z is in the first quadrant and has a large real part. Except for a missing common factor, the left-hand side in the particular case is half the Zenneck wave component of the wave function (as measured at the surface of the earth) for a vertical dipole centered on the interface between air and earth, the descending half of the series is the negative of the sky wave component of the wave function and the ascending part of the series is the convergent expansion for half the Zenneck wave component plus the sky wave component.

The suggestion immediately offered itself that (1) should hold for all values of ν, not just for $\nu = 1/2$.

If we assume for the moment that (1) holds for all ν, and differentiate with respect to z, we get

$$\begin{aligned} H_1^{(1)}\left(z(1-a)^{1/2}\right) &= \frac{-1}{(1-a)^{1/2}} \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu-1}^{(1)}(z) \\ &= \frac{-1}{(1-a)^{1/2}} \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} \left[\frac{2(n-\nu)}{z} H_{n-\nu}^{(1)}(z) - H_{n-\nu+1}^{(1)}(z)\right] \end{aligned}$$

* See John R. Carson, Transactions of this Society, vol. 31 (1929), pp. 782–792.