A REMARK ON METHOD IN TRANSFINITE ALGEBRA†

BY MAX ZORN‡

The theorems of Steinitz concerning algebraic closure and the degree of transcendence are barred, from the algebraic point of view, by the well-ordering theorem and its theory. We wish to show how, by introducing a certain axiom on sets of sets instead of the well-ordering theorem, one is enabled to make the proofs shorter and more algebraic. The proofs will be given in terms of the non-axiomatic standpoint of set theory.

DEFINITION 1. A set $\mathfrak{B}=\{B\}$ of sets B is called a *chain*, if for every two sets B_1, B_2, either $B_1 \supset B_2$ or $B_2 \supset B_1$.

DEFINITION 2. A set $\mathfrak{A}$ of sets A is said to be closed (right-closed), if it contains the union $\sum_{\mathfrak{B} \ni B} B$ of every chain $\mathfrak{B}$ contained in $\mathfrak{A}$.

Then our *maximum principle* is expressible in the following form.

(MP). *In a closed set $\mathfrak{A}$ of sets A there exists at least one, A^*, not contained as a proper subset in any other $A \in \mathfrak{A}$.*

APPLICATIONS. I. Let $\mathfrak{R}$ be a ring with a unity element 1; let $\mathfrak{A}$ be the set of all ideals $\mathfrak{a}$ (i) not containing 1 as an element, (ii) containing a certain ideal $\mathfrak{r} \neq \mathfrak{R}$. The set $\mathfrak{A}$ is obviously closed; the maximum principle implies the existence of a maximal ideal $\mathfrak{p}$ with $\mathfrak{r} \subseteq \mathfrak{p} \subset \mathfrak{R} \neq \mathfrak{p}$; this ideal is a prime ideal and the residue class ring $\mathfrak{R}/\mathfrak{p}$ is a *field*.

II. If k is a real field, that is, a field such that no sum of squares vanishes unless all the squares vanish, and K is an arbitrary algebraical extension, then the set of all real fields between k and K is closed, so the MP assures the existence of a *maximal* real field between k and K. In particular, if K is algebraically closed, we obtain a real closure of k.

III. Let K be an arbitrary field extension of k. A set of K-elements $\{a\}$ is said to be algebraically (respectively, linearly) independent, if no finite subset $a_1, a_2, a_3, \cdots, a_n$ satisfies an

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‡ Sterling Fellow.