DERIVATIVES OF HIGHER ORDER AS SINGLE LIMITS*

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1. *Introduction.* If a function has an nth derivative at a point, the value of this derivative may be calculated by taking the limit of a suitable quotient involving the nth difference, as the fundamental difference for the independent variable approaches zero. However, the limit of this quotient may exist without the function having the corresponding derivative. The principal result of this paper deals with the expression for unevenly spaced points, which reduces to the quotient mentioned above when the points are equidistant. It is shown that the existence of the limit of this expression, when the points close down in any way whatever to a fixed point, is a necessary and sufficient condition for the function to have an nth derivative throughout some neighborhood of the fixed point, continuous at the point. Some applications to finite Taylor developments are considered. A condition for the mere existence of the derivative is also given.

2. *Earlier Results.* A precise statement of the theorem on nth differences,† as proved by de la Vallée-Poussin, is that

$$\lim_{h\to 0} \frac{\Delta^n y}{h^n} = f^{(n)}(x_0), \tag{1}$$

if $y=f(x)$ has an nth derivative at x_0, where $\Delta^n y$ is the nth difference defined by the formulas

$$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x),$$
$$\Delta f(x) = f(x+h) - f(x).$$

The corresponding expression for unevenly spaced points is, except for a numerical factor, $n!$,

$$A_{n,k} = \sum_{j=1}^{n} \frac{f(x_j)}{(x_j - x_1)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_{n+1})}, \tag{2}$$

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† de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, vol. 1, 1921, p. 73.