DIFFERENTIATION OF SEQUENCES*

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Conditions under which one may differentiate term by term a convergent sequence of functions are to be found in the literature.† Some of these conditions are obtained as corollaries of theorems about termwise integration, and consequently contain in the hypothesis the assumption that the derived sequence converges. Without this assumption, sufficient conditions for termwise differentiability may be obtained from the fundamental theorem on reversing the order of iterated limits. This possibility is mentioned by Hobson (vol. 2, p. 336), but he does not obtain the theorems of the present paper.

By this means it may be shown that the equicontinuity of the functions of the derived sequence is sufficient for termwise differentiability, but the condition of equicontinuity is stronger than necessary for mere convergence, since it implies also that the derivatives are continuous and converge uniformly to a continuous function. This paper is concerned with weaker conditions, for which the names *normal*, *uniform*, *almost normal*, *almost uniform* are used. These conditions, like that of equicontinuity, are related also to the question of the compactness of sets of functions for different types of convergence. The procedure that will be followed is to state the principal lemma, define the different conditions, and then show their relations, first to compactness, and then to termwise differentiation.

LEMMA. *If the sequence of functions* (i) $\{f_n(x)\}$ *converges to* $f(x)$ *on the closed interval* $[a, b]$, *and the derivatives* $f_n'(x)$ *exist at the point* x_0 *of* $[a, b]$, *then a necessary and sufficient condition that the derivative* $f'(x_0)$ *exist at* x_0 *and be the limit of the sequence of derivatives* (ii) $\{f_n'(x_0)\}$ *is that for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that if* $|x - x_0| < \delta$, *then*

$$\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - f_n'(x_0) \right| < \epsilon \tag{1}$$

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† Hobson, *The Theory of Functions of a Real Variable*, 2nd ed., vol. 2, pp. 332–338.