BESICOVITCH ON ALMOST PERIODIC FUNCTIONS

Almost Periodic Functions. By A. S. Besicovitch. Cambridge, University Press, 1932. xiii+180 pp.

The seven years which elapsed between the time of creation of the theory of almost periodic functions by H. Bohr in 1925 and publication of Besicovitch's book, were years of remarkable development. The notion and properties of almost periodic functions, either in their initial, or in generalized form turned out to be of great importance in various fields of analysis, function theory, topology, and applied mathematics. The necessity of a monograph giving a concise and systematic exposition of the fundamentals of the theory of almost periodic functions was becoming more and more obvious. The task of writing such a monograph was an arduous one, but it is not astonishing that the author has completely succeeded in solving this difficult problem. The present monograph is but one of the numerous important contributions of the author to the theory of almost periodic functions for which he was awarded an Adams prize of the University of Cambridge in 1931. According to his own statement the author did not intend to give an encyclopaedic account of the theory; on the other hand, he was extremely painstaking in giving a clear and complete picture of the logical structure of the theory.

The book contains three chapters and an appendix between Chapters 2 and 3. Chapter 1, *Uniformly almost periodic functions* (66 pp.), is devoted to a discussion of continuous almost periodic functions as introduced initially by H. Bohr. The central point of the chapter is of course the proof of H. Bohr's "Fundamental Theorem," $\sum |A_n|^2 = M\{|f(x)|^2\}$. The initial proof of this theorem as given by H. Bohr was very complicated. Several other proofs were given later (Weyl, Wiener, de la Vallée-Poussin). The author gives the simplest of these proofs, which is due to de la Vallée-Poussin, and is based on a combination of ideas of Weyl and Bohr. Considerable attention is given also to the problem of summation of the Fourier series of an almost periodic function by means of sequences of partial sums, and also by means of the associated Fejér-Bochner trigonometric polynomials. A special paragraph is devoted to a discussion of arithmetical nature of translation numbers of an almost periodic function. The author considers the set $\overline{E}_\epsilon = \overline{E}(\epsilon; f(x))$ of all integers which are ϵ-translation numbers of $f(x)$, introduces the notion of almost periodicity of such sets, and proves that $\overline{E}_\epsilon$ is almost periodic for almost all values of ϵ. This theorem, due to the author jointly with H. Bohr, is interesting in itself and also plays an important role in the subsequent development of the theory of generalized almost periodic functions. The chapter closes with a rapid sketch of uniformly almost periodic functions of two independent variables.

Chapter 2, *Generalization of almost periodic functions* (56 pp.), treats of various generalizations of the notion of an almost periodic function. These generalizations, in the increasing order of their generality, are as follows: (S^p = Stepanoff) a.p. $\subset$ (W^p = Weyl) a.p. $\subset$ (B^p = Besicovitch-Bohr) a.p., $p \geqq 1$. They all include as the most special case the (U = uniformly) a.p. functions of Bohr. While (S^p) and (W^p) a.p. functions represent a "natural" generaliza-