A THEOREM ON ANALYTIC FUNCTIONS OF A REAL VARIABLE

BY R. P. BOAS, JR.

1. *Introduction*. Let $f(x)$ be a function of class C^∞ on $a \leqq x \leqq b$. At each point x of $[a, b]$ we form the formal Taylor series of $f(x)$,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} (t - x)^k.$$

This series has a definite radius of convergence, $\rho(x)$, zero, finite, or infinite, given by $1/\rho(x) = \overline{\lim}_{k\to\infty} |f^{(k)}(x)/k!|^{1/k}$. The function $f(x)$ is said to be analytic at the point x if the Taylor development of $f(x)$ about x converges to $f(t)$ over a neighborhood $|x-t| < c$, $c > 0$, of the point; $f(x)$ is analytic in an interval if it is analytic at every point of the interval.

Pringsheim stated the following theorem.*

THEOREM A. *If there exists a number $\delta > 0$ such that $\rho(x) \geqq \delta$ for $a \leqq x \leqq b$, $f(x)$ is analytic in $[a, b]$.*

However, Pringsheim's proof of the theorem is not rigorous. The purpose of this note is to establish this theorem, and, in connection with the proof, a companion theorem of considerable interest in itself.

THEOREM B. *If $\rho(x) > 0$ for $a \leqq x \leqq b$ (that is, if the Taylor development of $f(x)$ about each point converges in some neighborhood of the point), the points at which $f(x)$ is not analytic form at most a nowhere dense closed set.*

Theorem B is, in a certain sense, the best possible, since by a theorem of H. Whitney† there exist functions satisfying the

* A. Pringsheim, *Zur Theorie der Taylor'schen Reihe und der analytischen Funktionen mit beschränkten Existenzbereich*, Mathematische Annalen, vol. 42 (1893), p. 180.

† H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of this Society, vol. 36 (1934), pp. 63–89. I am indebted to Dr. Whitney for calling my attention to this paper.