# A NOTE ON TAYLOR'S THEOREM

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Let the function $f(x)$ be such that $f^{(n)}(a) \equiv d^n f(x)/dx^n$ at $x=a$ exists; then, for $|h|$ sufficiently small, we can write

$$(1)\ f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + w(a, h).$$

It is well known that $w(a, h) = o(h^n)$ as $h \to 0$,* and the more precise result that $|w(a, h)| \leqq |h^n| v(a, h)$, where $v(a, h)$ is the least upper bound for $0 < |t| < |h|$ of

$$\left| \frac{f^{(n-1)}(a+t) - f^{(n-1)}(a)}{t} - f^{(n)}(a) \right|$$

is given by S. Pollard.†

In this note we are concerned primarily with the behavior, as $h \to 0$, of derivatives with respect to $h$ of the function $w(a, h)$. The point $a$ being fixed, we designate the $i$th such derivative, $i \geqq 0$, by $d^i w(a, h)/dh^i$. Our theorem, a generalization of Pollard's theorem, is given below.

THEOREM. *If $f(x)$ is such that $f^{(n)}(a)$ exists, then for $i=0, 1, 2, \cdots, n-1$, and $|h|$ sufficiently small*

$$\left| \frac{d^i}{dh^i} w(a, h) \right| \leqq \frac{|h^{n-i}|}{(n-i)!} v(a, h).$$

PROOF. Since

$$\frac{d^i}{dt^i} f(a+t) \equiv \frac{d^i}{dx^i} f(x) \Big]_{x=a+t} \equiv f^{(i)}(a+t),$$

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* See E. W. Hobson, *The Theory of Functions of a Real Variable*, vol. 1, 3d ed., pp. 368–370. We use here the more restrictive of the two definitions given by Hobson for $f^{(n)}(x)$. The existence of $f^{(n)}(a)$ then insures the existence and continuity in an open interval containing $a$ of all derivatives of lower order.

† S. Pollard, *On the descriptive form of Taylor's theorem*, Cambridge Philosophical Society Proceedings, vol. 23 (1926–27), pp. 383–385. Pollard's proof seems only to establish the less sharp result $|w(a, h)| \leqq n |h^n| v(a, h)$.