these questions, will be awaited with great interest. It may be that still other points of view will be necessary to complete the foundations of mathematics so well begun by the Hilbert proof-theory.

SAUNDERS MACLANE

SZEGÖ ON JACOBI POLYNOMIALS

Asymptotische Entwicklungen der Jacobischen Polynome. By Gabriel Szegö. Schriften der Königsberger Gelehrten Gesellschaft. Jahr 10, Naturwissenschaftliche Klasse, Heft 3, 1933, pp. 35–111 (1–77).

One of the interesting features in the development of analysis in the twentieth century is the remarkable growth, in various directions, of the theory of orthogonal functions. Two brilliant achievements on the threshold of this century—Fejér's paper on Fourier series and Fredholm's papers on integral equations—have been acting as a powerful inspiring source of attraction, inviting analysts to delve deeper into the theory of orthogonal functions and their applications. First come, due to their simplicity, the trigonometric functions $\{\sin mx, \cos mx\}$ which serve as a yardstick for orthogonal functions in general. Next we may consider orthogonal polynomials, of which Jacobi polynomials are a special case.

Let us recall the general definition of orthogonal polynomials. A weight-function $p(x)$, non-negative in a given interval (a, b), finite or infinite, and such that all "moments" $\int_a^b p(x)x^r dx = \alpha_r$ exist, $(r = 0, 1, 2, \cdots)$, with $\alpha_0 > 0$, gives rise to a unique system of orthogonal and normal polynomials $\phi_n(x) = a_n x^n + \cdots$, $(n = 0, 1, \cdots; a_n > 0)$, so that

$$\begin{aligned}(1)\qquad \int_a^b p(x)\phi_m(x)\phi_n(x)dx &= 0,\ (m \neq n),\\ &= 1,\ (m = n),\quad (m, n = 0, 1, \cdots).\end{aligned}$$

On the basis of (1), we obtain the following expansion of an "arbitrary" function:

$$(2)\qquad f(x) \sim \sum_{n=0}^{\infty} f_n\phi_n(x),\quad \text{with}\quad f_n = \int_a^b p(x)f(x)\phi_n(x)dx,$$

and this constitutes the most interesting and important application of the polynomials $\phi_n(x)$ in analysis, as well as in mathematical physics, mathematical statistics, etc.

The oldest and best known are Legendre polynomials, derived from (1) with (a, b) finite, say $(-1, 1)$, and $p(x) \equiv 1$. Their direct generalization are Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$: $(a, b) = (-1, 1)$, $p(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$. In case of an infinite interval, the most important are the polynomials of Laguerre: $(a, b) = (0, \infty)$, $p(x) = x^\alpha e^{-x}$, $\alpha > -1$, and those of Hermite: $(a, b) = (-\infty, \infty)$, $p(x) = e^{-x^2}$. These four kinds of orthogonal polynomials constitute what may be considered as one single family of "classical" polynomials, where Jacobi polynomials, from many points of view, represent the most typical member. In fact, by assigning to α, β certain finite or limiting values, we get Legendre polynomials $(\alpha = \beta = 0)$, trigonometric polynomials $(\alpha = \beta = -1/2)$,