HILBERT-BERNAYS ON PROOF-THEORY

Grundlagen der Mathematik, Volume I. By D. Hilbert and P. Bernays. (Grundlehren der Mathematischen Wissenschaften, Volume XL.) Berlin, Springer, 1934. xii+471 pp.

This is undoubtedly the most important book on the foundations of mathematics since Whitehead and Russell's *Principia Mathematica*, for it offers an authoritative formulation of the famous Hilbert *Proof-theory*. All the recent work in the foundations has been dominated by the discovery of contradictions in the body of mathematics, especially in Cantor's *Mengenlehre*. There have been many attempts to avoid this difficulty by using axiom systems so limited that the known contradictions could not arise. Hilbert, however, planned a direct attack on the difficulty: an attempt to prove that, in a suitably limited system, no *new* contradiction could ever arise. Bernays here discusses the cases in which Hilbert's plan has succeeded. In brief, the Hilbert school has developed a powerful and fascinating method for investigating mathematical proofs and has shown by these methods that a large part of elementary number theory is *consistent* (free from contradiction). However, the extension to more complicated branches of mathematics has met with serious obstacles.

How is it possible to show that a mathematical system is consistent? Only by means of a thoroughgoing formalization of the axioms and proofs of that system. In other words, the logical methods usually used uncritically in carrying out a mathematical proof must themselves be subjected to mathematical formulation. This is possible by means of the calculus of propositions, which was developed by Peano and by Russell and Whitehead. In this calculus, all the axioms of logic and mathematics can be precisely and symbolically expressed. Furthermore, the operations of logic are all reduced to a few simple, mechanical rules of procedure. A proof must thus start with one or more known axioms, and must proceed step by step, each step following some one of the mechanical rules. Any formal proof is thus finite and combinatorial in character, and hence the possibility that some proof might lead to a contradiction can be investigated by combinatorial methods. This is Hilbert's plan of attack.

But this finite analysis of formal proofs must itself be mathematical and so must itself involve proofs. These latter proofs belong to metamathematics—they are not the mathematics to be investigated; they are rather the tools of the investigation. For example, any general study of proofs will need some sort of complete induction on the number of steps in a proof. This process of induction, together with the other tools needed in the investigation, is essentially *finite* in character. Bernays has explained excellently exactly wherein this finiteness consists. Roughly speaking, finite arguments about numbers are those which can be grasped perceptually (that is, which are *anschaulich überblickbar*). In particular, the existence of a number with some property has a finite meaning only when there is a definite method whereby some such number can be constructed. In this respect, finite theorems are subject to the intuitionistic logic of Brouwer. This means that the tools of proof-theory are to be finite methods which are themselves clearly consistent.