ON THE rTH DERIVED CONJUGATE FUNCTION*

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1. *Introduction.* We assume throughout this note that the function $f(x)$ is Lebesgue integrable on $(-\pi, \pi)$ and of period 2π; then the series

$$\sum_{n=1}^{\infty} (- b_n \cos nx + a_n \sin nx), \tag{1}$$

where a_n and b_n are the Fourier coefficients, is known as the conjugate Fourier series.

It is customary to associate with the series (1) as sum or "conventionalized" sum the conjugate function of $f(x)$ which is either the limit

$$\tilde{f}_1(x) \equiv \frac{-1}{2\pi} \lim_{\epsilon\to 0} \int_{\epsilon}^{\pi} \{f(x+s) - f(x-s)\} \operatorname{ctn} \frac{s}{2}\, ds, \tag{2}$$

or the equivalent limit

$$\tilde{f}_2(x) \equiv \frac{-1}{\pi} \lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} \frac{f(x+s) - f(x-s)}{s}\, ds; \tag{3}$$

and to associate with the first derived series of the series (1) the first derived conjugate function which is either the limit

$$\tilde{f}_1'(x) \equiv \frac{-1}{4\pi} \lim_{\epsilon\to 0} \int_{\epsilon}^{\pi} \{f(x+s) + f(x-s) - 2f(x)\} \csc^2 \frac{s}{2}\, ds, \tag{4}$$

or the equivalent limit

$$\tilde{f}_2'(x) \equiv \frac{-1}{\pi} \lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} \frac{f(x+s) + f(x-s) - 2f(x)}{s^2}\, ds.\dagger \tag{5}$$

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† For a proof of the equivalence of (2) and (3) see G. H. Hardy and J. E. Littlewood, *The allied series of a Fourier series*, Proceedings of the London Mathematical Society, vol. 24 (1925), pp. 211–246 (p. 221). A proof of the simultaneous existence of (4) and (5) is given by A. H. Smith, *On the summability of derived series of the Fourier-Lebesgue type*, Quarterly Journal of Mathematics, (Oxford Series), vol. 4 (1933), pp. 93–106 (p. 106); that they are equivalent follows from Theorem 1 of this note.