A POLAR REPRESENTATION OF SINGULAR MATRICES

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Let $A=(a_{ij})$, $(i=1, 2, \cdots, m; j=1, 2, \cdots, n)$, be a matrix of m rows and n columns, whose elements a_{ij} are complex numbers. It has been shown† that, if $m=n$ and A is non-singular, $A=P_1U=UP_2$, where U is a unitary matrix, while P_1 and P_2 are positive definite hermitian matrices. Moreover in such a polar representation of A, as it has been called, the matrices P_1, P_2, and U are uniquely determined. We shall show that, if $m=n$ and the rank of A is $r<n$, $A=P_1U=UP_2$, where P_1 and P_2 are uniquely determined positive hermitian matrices of rank r and U is unitary but no longer unique. Any such representation of course is impossible if $m\neq n$, as by definition both hermitian and unitary matrices are square, but it will be shown that somewhat analogous results exist in this case as well.

As is customary we shall denote the conjugate transposed of A by $A^*=(a_{ij}^*)$, where $a_{ij}^*=\bar{a}_{ji}$, the complex conjugate of a_{ji}. We shall use this notation, even if A is a vector, that is, a matrix of one row, so that in this case AA^* will simply denote the norm of the vector A. For the sake of brevity we shall use the notations E_j for the unit matrix of order j and $0_{i,j}$ for the zero matrix of i rows and j columns.

The matrix $N_1=AA^*$ is a square matrix of order m and the matrix $N_2=A^*A$ is a square matrix of order n, and since $N_1=N_1^*$ and $N_2=N_2^*$, both of these matrices are hermitian. Moreover, if the rank of A is r, the rank of N_1 is r and so is the rank of N_2. For, if K is the rth compound‡ of A, at least one element k_{ij} of K is different from zero. The element in the ith place of the leading diagonal of the product matrix KK^* is $\sum_t k_{it}\bar{k}_{it}$, which is a positive real number, since k_{ij} is not zero. Accordingly there is at least one r-rowed determinant of N_1

† L. Autonne, Bulletin de la Société Mathématique, vol. 30 (1902), pp. 121–134. A. Wintner and F. D. Murnaghan, *On a polar representation of non-singular matrices*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 676–678.

‡ Turnbull and Aitken, *The Theory of Canonical Matrices*, p. 27.