MECHANICAL INVARIANTS OF THE SWEEPING-OUT PROCESS*

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In this paper we prove the following theorem.

THEOREM. *If a general bounded distribution of positive mass in a closed connected region R is swept out on a surface S entirely enclosing R in its interior, then the center of gravity and the principal axes are invariants for the sweeping-out transformation.*

Let the distribution be given by $\Phi(e)$, which means the mass associated with the point set e. Then the potential is

$$V(M) = \int_R \frac{1}{\overline{MP}}\, d\Phi(e_P).$$

The coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the center of gravity of the distribution are, respectively,

$$\frac{1}{\Phi(R)}\int_R x_P d\Phi(e_P), \qquad \frac{1}{\Phi(R)}\int_R y_P d\Phi(e_P), \qquad \frac{1}{\Phi(R)}\int_R z_P d\Phi(e_P).$$

We have the following lemma.

LEMMA. *If Φ is such that a density ρ exists and $\nabla^2 V = -4\pi\rho$ is satisfied everywhere, then the sweeping-out on a level surface Σ of V entirely including R leaves the center of gravity and the principal axes invariant.*

The surface Σ is formed by setting $V=\delta>0$. Let P_0 be the center of gravity of the distribution Φ. Let R_0 be the lower bound of the radii of all spheres containing R with center P_0. If M_0 is a large integer, P on Σ, and $\delta=\Phi(R)(M_0R_0)^{-1}$, then we shall have $R_0(M_0-1)\leqq\overline{PP}_0\leqq R_0(M_0+1)$. Hence Σ lies in the spherical shell whose center is the point P_0 and whose bounding radii are $R_0(M_0-1)$ and $R_0(M_0+1)$.

Let u be any function harmonic in a closed region v containing P_0 whose boundary is the level surface Σ. Then, applying Green's Theorem, we have

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