THE CATEGORY OF THE CLASS LIP (α, p)

BY E. S. QUADE

A function $x(s)$ is said to belong to the class Lip(α, p) on the interval (a, b) provided

$$\|x(s+h) - x(s)\| \equiv \left(\int_a^b |x(s+h) - x(s)|^p ds\right)^{1/p} = O(h^\alpha),$$

where $0 < \alpha \leqq 1$.

There exist continuous functions which belong to no class Lip(α, p). Indeed if $x(s) \subset \mathrm{Lip}(\alpha, p)$, then the Fourier coefficients of $x(s)$, a_n, b_n, are $O(n^{-\alpha})$. Now a continuous function may be constructed* such that $|a_{n_i}| > 1/\log n_i$ for an infinite set of values $\{n_i\}$. Then for such a function

$$\frac{|a_{ni}|}{n_i^{-\alpha}} > \frac{n_i^\alpha}{\log n_i} \neq O(1),$$

that is, $a_n \neq O(n^{-\alpha})$ and hence the continuous function with the Fourier coefficients a_n belongs to no class Lip (α, p).

We prove the following theorem.

THEOREM. *The subset E of L_p, $p \geqq 1$, which is* $\sum$ Lip(α, p) *for $0 < \alpha \leqq 1$, is of the first category in L_p.*

We employ a method of proof used by S. Banach.† We take the interval (0, 1) as the fundamental interval and assume the functions to be periodic with the period one. Let E_{nm} be the set of all $x(s) \subset L_p$ such that

$$\int_0^1 |x(s+h) - x(s)|^p ds \leqq n^p |h|^{p/m}, \qquad (n, m = 1, 2, \cdots).$$

The sets E_{nm} are closed. For, let $x_i(s) \rightarrow x_0(s)$ in L_p. Set

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* W. Randels, *A remark on Fourier series of continuous functions*, American Mathematical Monthly, vol. 40 (1933), pp. 97–99. See also an article by O. Szász, to appear soon in the same journal.

† *Über die Baire'sche Kategorie gewisser Funktionenmengen*, Studia Mathematica, vol. 3 (1931), pp. 174–179.