ON APPROXIMATION TO AN ANALYTIC FUNCTION BY POLYNOMIALS*

BY O. J. FARRELL

1. *Introduction.* If the function $f(z)$ is analytic in the interior R of a Jordan curve C and continuous in the corresponding closed region $\overline{R}$, then in that closed region $f(z)$ can be uniformly expanded in a series of polynomials in z. This result is due to Walsh,† who recently encouraged the writer to study the problem of polynomial approximation to a function $f(z)$ analytic interior to C but not necessarily continuous in $\overline{R}$ or even bounded in R. Professor Walsh suggested (1) that if $f(z)$ is bounded in R, there might exist polynomials $p_n(z)$, $(n=1, 2, \cdots)$, which approximate to $f(z)$ in R in such a way that the limit as $n\rightarrow\infty$ of the least upper bound of $|p_n(z)|$ for z in R does not exceed the least upper bound of $|f(z)|$ for z in R; (2) that if the double integral over R of $|f(z)|^p$, $(p>0)$, exists, there might be a sequence of polynomials $\{p_n(z)\}$ which approximate to $f(z)$ in R so that the limit as $n\rightarrow\infty$ of the double integral over R of $|f(z)-p_n(z)|^p$ is zero. It is the purpose of the present note to show that the existence of such approximating polynomials can be established in both cases by a method due to Carleman‡ not only for a Jordan region but also for regions of somewhat more general boundary.

It is a well known characteristic of a finite simply connected region R bounded by a Jordan curve C that the totality of points of the extended plane not belonging to $R+C$ form a single region (also simply connected) whose boundary is precisely C. But the class of regions characterised by this property includes not only all finite Jordan regions but regions of more general boundary as well.§ And it has been found that the results sug-

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† Walsh, Mathematische Annalen, vol. 69 (1926), p. 430.

‡ Carleman, Arkiv för Matematik, Astronomi och Fysik, vol. 17 (1923), pp. 1–30, §1.

§ For an example of such a region with a boundary which is not a Jordan curve see, for instance, Carathéodory, Mathematische Annalen, vol. 73, pp. 305–370, §35.