TRANSFINITE SUBGROUP SERIES*

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1. *Summary*. This note contains a proof that the Theorem of Jordan-Hölder can be extended to the case of any series of normal subgroups or, more generally, to the case of what we shall call "T-invariant" subgroups well-ordered in the direction of increasing subgroups. An example is given showing that the replacement of "increasing" by "decreasing" in the preceding sentence renders the proposition false.

Finally, the situation as regards the subgroup-series of compact topological groups homeomorphic with subsets of Cartesian n-space is clarified by two superficial observations.

2. *Definitions and Notation*. Let G be any group; H and K any two subgroups of G. We shall write $H \cap K$ for the *meet* or cross-cut of H and K, and $H \cup K$ for the subgroup generated by the *join* of H and K. The statements $H<K$ and $K>H$ mean that H is contained in, but is different from, K; $H \nless K$ and $K \ngtr H$ mean that $H<K$ is false. The statement $H \supset K$ means H includes K.

Now let A be the group of all automorphisms, and A_I the subgroup of the inner automorphisms of G, and let T be any subgroup of A containing A_I. The subgroup H will be called *T-invariant* if and only if it is carried into itself under every automorphism of T. It is certain that any T-invariant subgroup is normal.

By a *T-series* of G we shall mean† any set Σ of T-invariant subgroups T_i of G with the two properties:

(i) If $i \neq j$, then either $T_i<T_j$ or $T_i>T_j$.

(ii) To every T-invariant subgroup X of G corresponds a $T_i \in \Sigma$ such that $T_i \nless X$ and $T_i \ngtr X$.

By a *well-ordered ascending* (*well-ordered descending*) T-series of G is meant one in which every subset has a least (greatest) term.

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† The cases $T=A_I$ and $T=A$ yield under these definitions normal subgroups and chief series, and characteristic subgroups and characteristic series. The cases $A_I<T<A$ yield generalizations.