THE CONVERSE OF WARING'S PROBLEM

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1. *Introduction.* In the most general Waring problem we are given a set of integers $\geqq 0$ and seek k such that every integer (or every sufficiently large integer) is a sum of k numbers of the set. We then call the set a k-set. In the converse problem, k is given and we seek all k-sets. There exist infinitely many k-sets; for example, (1) and (2).

In case every integer $\geqq 0$ is a sum of k numbers of a set, we call the latter a universal k-set. It must contain 0 and 1. By way of introduction, we construct some universal 2-sets.

I. As the nth element of the set choose the least integer which is not a sum of any two of the first $n-1$ elements. The set is composed of 0 and all positive odd integers.

II. The set with 0, 1, 2 and later elements chosen as in I is composed of 0, 1, $2+3x$, $(x=0, 1, \cdots)$. It is a universal 2-set.

III. After 0 and 1 choose the nth element as in I when n is odd, but subtract 1 from the least when n is even. We get the universal 2-set composed of 0, 1, $3+6x$, $4+6x$, $(x=0, 1, \cdots)$.

IV. After 0 and 1 employ blocks of three elements. Those in a block are odd (least as in I), odd (least), even (least less 1). We get the universal 2-set 0, 1, $3+10x$, $5+10x$, $6+10x$, $(x=0, 1, \cdots)$.

V. Our aim here is to construct a bizarre 2-set. After 0, 1 employ blocks of 2, 3, 4, $\cdots$ elements, where the last element of a block is even and the others are all odd, while the nth element is either the least or 1 less than the least integer which is not a sum of any two of the first $n-1$ elements. We get the set 0, 1, 3, 4, 9, 11, 16, 21, 23, 27, 28, 33, 35, 39, 41, 46, 53, 59, 65, 71, 77, 82, 83, 89, 95, 97, 101, 107, 114, 119, 125, 127, 133, 139, 145, 151, 156, 163, 169, 175, 181, 187, 193, 199, 205, 212, 217, 219, 225, 231, 237, 243, 249, 255, 261, 266, 267, 273, 279, 285, 291, 297, 303, 309, 311, 317, 322, 329, etc.

Unlike I–IV, there is apparently no simple independent definition of this 2-set. If we take the first elements of the successive blocks and form their differences of the second order,