NOTE ON THE ITERATION OF FUNCTIONS OF ONE VARIABLE*

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1. *Introduction*. Let $E(x)$ be a real-valued function of the real variable x for some specified range, and let

$$E_0(x) = x,\ E_1(x) = E(x),\ \cdots,\ E_{n+1}(x) = E(E_n(x)),\ \cdots$$

represent its successive iterations. The interpolation problem of defining $E_n(x)$ for non-integral values of n was discussed some time ago by A. A. Bennett,† who reduced it formally to the solution of the functional equation

$$\psi(x + 1) = E(\psi(x)). \tag{1}$$

For if $\psi(x)$ satisfies (1) and if n is any positive integer,

$$\psi(x + n) = E_n(\psi(x)). \tag{2}$$

Hence on writing $\psi^{-1}(x)$ for x, where $\psi^{-1}(x)$ denotes an inverse of the function $\psi(x)$, we obtain the formula

$$E_n(x) = \psi(\psi^{-1}(x) + n), \tag{3}$$

defining $E_n(x)$ for a continuous range of values of n.

In this note, I propose to give an entirely elementary explicit solution to this problem of interpolation for all functions $E(x)$ subject to the following three conditions:‡

(a). *$E(x)$ is a real, continuous, single-valued function of the real variable x in the range $a \leqq x < \infty$.*

(b). *$E(x) > x$ for all $x \geqq a$.*

(c). *$E(x') > E(x)$ if $x' > x \geqq a$.*

We may remark that the functional equation (1) is merely another form of a famous equation studied by Abel,§

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† In two papers in the Annals of Mathematics, (2), vol. 17 (1915–16), pp. 74–75 and pp. 23–60. This second paper contains references to the earlier literature. A. Korkine (Bulletin des Sciences Mathématiques, (2), vol. 6 (1882), pp. 228–242) seems to have been the first to consider this problem.

‡ These conditions are all satisfied by $E(x) = e^x$, the particular case discussed by Bennett in the first paper cited.

§ Works, vol. II, *Posthumous Papers*, 1881, pp. 36–39.