A METRICAL PROPERTY OF POINT-SET TRANSFORMATIONS*

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It is the purpose of this note to prove a theorem concerning distances between points of a metric space‡ under transformations of the space into itself. The distance between two points p and q of a metric space is denoted by $d(p, q)$. The point into which a point p is taken by n applications of a transformation T is denoted by p_n, that is, T takes p to p_1, p_1 to p_2, and so on. If $d(p, q)$ is less than, greater than, or equal to $d(p_1, q_1)$, T is said, respectively, to increase, decrease, or to leave invariant the distance between p and q. The theorem of this note shows that any continuous transformation of a suitably restricted metric space into itself leaves invariant the distance between some two distinct points of the space. In particular, it follows from the theorem that any continuous transformation of a closed or open n-sphere into itself leaves the distance between two distinct points of the sphere invariant. The theorem also applies to spherical surfaces.

It should be observed that in the following lemma the transformation is not assumed to be continuous as it is in the theorem. In both cases the phrase "into itself" means that the space goes into the whole of itself and not into a proper subset of itself.

LEMMA. *If E is a conditionally compact metric space and if T is any one-to-one transformation of E into itself which increases the distance between two points of E, then T must also decrease the distance between two points of E.*

Let p and q be two points of E whose distance is increased by T, that is, $d(p, q) < d(p_1, q_1)$. The proof will be made by contradiction. Suppose that the lemma is false and that, for every two points x and y of E, we have $d(x, y) \leqq d(x_1, y_1)$.

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‡ For definitions of metric spaces and terms concerning them see Hausdorff, *Mengenlehre*, 1927.