ON THE LATTICE THEORY OF IDEALS†

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1. *Outline.* The ideals of any ring define, relative to g.c.f. and l.c.m., a combinatorial system having properties which we shall presently define as characterizing *B-lattices.*

In this article we shall first develop some new properties of B-lattices as abstract systems; the main results of this part of the work find expression in Theorems 1–5. Then we shall apply this theory and some older results to the ideals of commutative rings R which possess a principal unit l and satisfy the Basis Theorem. In addition to developing the known theory of *einartig* ideals by combinatory methods, we give a necessary and sufficient condition that the *B-lattice* defined by the ideals of R should be isomorphic with a *ring* of point sets in the sense of Hausdorff.‡

2. *Notation*; *Lattice Algebras.* We shall in general use capital letters to denote systems, and small letters for elements. $a \in A$ will mean "a is an element of the system A"; $B \subset A$ will mean "$b \in B$ implies $b \in A$"; $B < A$ will mean $B \subset A$ but $B \neq A$.

By a *lattice algebra* will be meant any system L which satisfies the following postulates:

(L1). Any $a \in L$ and $b \in L$ determine a unique "*join*" $a \cap b \in L$ and a unique "*meet*" $(a, b) \in L$.

(L2). $a \cap b = b \cap a$ and $(a, b) = (b, a)$ for any $a \in L$ and $b \in L$.

(L3). $a \cap (b \cap c) = (a \cap b) \cap c$ and $(a, (b, c)) = ((a, b), c)$ for any $a \in L$, $b \in L$, and $c \in L$.

(L4). $a \cap (a, b) = a$ and $(a, a \cap b) = a$ for any $a \in L$ and $b \in L$.

From (L1)–(L4) follow $a \cap a = (a, a) = a$. Moreover $a \cap b = b$ is equivalent to $(a, b) = a$; in this case we write $a \subset b$ or $b \supset a$, and $a \subset b$ taken with $b \subset c$ implies $a \subset c$. Moreover, $a < b$ means $a \subset b$ but $a \neq b$, while "b covers a" means $a < b$, but that no $x \in L$ satisfies $a < x < b$.

The reader may find it helpful to regard lattices as distorted

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‡ Hausdorff, *Mengenlehre*, 1927, p. 77.