ON DIRECT PRODUCT MATRICES†

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1. *Introduction.* If $A=(a_{ij}), (i=1, 2, \cdots, m; j=1, 2, \cdots, n)$, is an $m\times n$ matrix and B is a $p\times q$ matrix, then the matrix, $P=(a_{ij}B)=A\langle B\rangle$, of order $mp\times nq$, whose elements occur in mn blocks, $a_{ij}B$, is the direct product of A and B.‡

In the present paper we determine the elementary divisors of $A\langle B\rangle-\lambda I$ and of $\rho A\langle I\rangle+\sigma I\langle B\rangle-\lambda I$, where ρ and σ are scalar constants, and where the elementary divisors of $A-\lambda I$ and of $B-\lambda I$ are known. Finally, §3 takes up the discussion of the linear matrix equation

$$A_1X_1B_1 + A_2X_2B_2 + \cdots + A_rX_rB_r = C.$$

The reduction of this equation to an equation whose solution is known is accomplished by means of direct product matrices and thus perfects a procedure first noted by MacDuffee.§

2. *On Elementary Divisors of* $A\langle B\rangle-\lambda I$ *and of* $\rho A\langle I\rangle+\sigma I\langle B\rangle-\lambda I$. In this section it will be convenient to indicate the order of a matrix by subscripts; thus $A_{\alpha,\beta}$ is an $\alpha\times\beta$ matrix, B_p is a square matrix of order p, and I_p is the unit matrix of order p. Matrices will be designated throughout by capitals, whereas lower case letters will be employed to denote scalars, such as parameters, constants, and the elements of matrices. Moreover, all scalars will be regarded as belonging to the complex number field. Hence, for our purpose, $(a_{ij}B)=(Ba_{ij})=A\langle B\rangle=\langle B\rangle A$.

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† Presented to the Society, April 6, 1934.

‡ Zehfuss, *Ueber eine gewisse Determinante*, Zeitschrift für Mathematik und Physik, 3te Jahrgang (1858), pp. 298–301, was perhaps the first to study determinants of this form. Rutherford, *On the condition that two Zehfuss matrices be equal*, this Bulletin, vol. 39 (1933), pp. 801–808, called P the Zehfuss matrix of A and B and devised the notation here employed. Dickson, *Algebras and their Arithmetics*, 1923, p. 119, and MacDuffee, *The Theory of Matrices*, 1933, p. 81, employ the term *direct product* to designate P. The reader is also referred to the latter treatise for a more complete discussion of direct product matrices.

§ MacDuffee, loc. cit., p. 89.