DYNAMICAL TRAJECTORIES AND CURVATURE TRAJECTORIES*

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1. *Introduction.* In this paper we compare two important types of triply infinite families of plane curves, *dynamical* families and *curvature* families. Both types are projectively invariant, so that our subject belongs to the *projective differential geometry* of systems of curves.

A family of *dynamical trajectories* consists of the ∞^3 possible paths of a particle moving in a general field of force, initial position and velocity being arbitrary. If $\phi(x, y)$ and $\psi(x, y)$ are the components of force, the equations of motion are

$$(1') \qquad \ddot{x} = \phi(x, y), \qquad \ddot{y} = \psi(x, y).$$

The differential equation of the trajectories† is found by eliminating the time from (1′):

$$(1) \quad (\psi - y'\phi)y''' = \{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\} y'' - 3\phi y''^2.$$

To define *curvature trajectories* we start with an arbitrary doubly infinite family of curves, that is, a general differential equation of second order:

$$(2') \qquad y'' = F(x, y, y').$$

A *curvature trajectory* of this family is a curve which is drawn so as to have at each point c times the curvature of the member of the family to which it is tangent at that point, c remaining constant along the trajectory. For a given value of c there will be a set of ∞^2 trajectories, (one in each direction through each point). By varying c we obtain ∞^1 such sets. Hence a given doubly infinite family (2′) generates a triply infinite family of curvature trajectories.

* Presented to the Society, April 26, 1919, under the title *A characteristic property of central forces*. See abstract, this Bulletin, vol. 25 (1919), p. 443.

† E. Kasner, *The trajectories of dynamics*, Transactions of this Society, vol. 7 (1906), pp. 401–424. Also *Differential-Geometric Aspects of Dynamics*, Princeton Colloquium Lectures on Mathematics, 1913, especially Chapters 1 and 3, where some of the properties are stated in projective language. A new edition, published by this Society, has just appeared.