ON CONVERGENCE IN VARIATION*

BY C. R. ADAMS AND J. A. CLARKSON

1. *Introduction.* Certain questions concerning functions $f(x, y)$ of bounded variation naturally lead one to consider a sequence of functions $f_n(x)$, $(n=1, 2, 3, \cdots)$, defined on an interval† (a, b) and satisfying the following conditions: $f_n(x)$ tends to a limit function $f_0(x)$ of bounded variation; the total variation $T_a^b(f_n)$ of $f_n(x)$ on (a, b) tends to the total variation $T_a^b(f_0)$ of $f_0(x)$ on (a, b).‡ The notation $f_n(x) \xrightarrow{v} f_0(x)$ will frequently be employed to describe this situation, which has already received attention from Buchanan and Hildebrandt.§ All of the theorems which we are about to establish are valid when a set of functions $f(x, \lambda)$ corresponding to a set of values λ having λ_0 as a limit is considered, with $f(x, \lambda) \to f_0(x)$ as $\lambda \to \lambda_0$ over the set.

2. *Preliminary Theorems.* Let $P_n[N_n]$ denote the total positive [negative] variation of $f_n(x)$ on (a, b), $(n=0, 1, 2, \cdots)$; then we have the following theorem.

THEOREM 1. *The relations* $f_n(a) \to f_0(a)$, $f_n(b) \to f_0(b)$, *and* $T_a^b(f_n) \to T_a^b(f_0)$ *imply* $P_n \to P_0$ *and* $N_n \to N_0$.

This follows at once by writing

$$f_n(b) = f_n(a) + P_n - N_n, \qquad (n=0, 1, 2, \cdots).$$

THEOREM 2. *The relation* $f_n(x) \to f_0(x)$ *on* (a, b) *implies*

$$\liminf_{n \to \infty} T_a^b(f_n) \geqq T_a^b(f).$$

This may easily be proved directly or by aid of the well

* Presented to the Society, December 27, 1933.

† The *closed* interval is always to be understood.

‡ It may be of interest to note that $T_a^b(f(x))$ is a *semi-linear* operation in the sense that we have $T_a^b(f(x)+g(x)) \leqq T_a^b(f(x)) + T_a^b(g(x))$ and $T_a^b(cf(x)) = |c| T_a^b(f(x))$ for c constant.

§ Buchanan and Hildebrandt, *Note on the convergence of a sequence of functions of a certain type*, Annals of Mathematics, (2), vol. 9 (1908), pp. 123–126. This paper will be referred to as BH. The symbol $\xrightarrow{v}$ may be read "*converges in variation.*"