THE WORK OF POINCARÉ ON DIFFERENTIAL EQUATIONS

Oeuvres de Henri Poincaré publiées sous les auspices de l'Académie des Sciences par Paul Appell. Tome I publié avec la collaboration de Jules Drach. Paris, Gauthier-Villars, 1928. cxxix+382 pp.

This is the second volume of the Collected Works of Henri Poincaré to appear, the first being volume II, which was published in 1916.* The earlier volume contained his contributions in the general field of automorphic functions, while the new one contains those concerned primarily with ordinary and partial differential equations, and linear difference equations. The order of the material of the Collected Works is to be that contained in Poincaré's own *Analyse* of his papers, and it is decidedly helpful to the reader to find the relevant part of the *Analyse* concerning the material of these first two volumes at the beginning of volume I. A similar plan is to be followed with subsequent volumes.

The starting point of Poincaré's extraordinary mathematical activities is to be found in his *Note sur les propriétés des fonctions définies par les équations différentielles*, published in 1878 in the Journal de l'École Polytechnique. The usual existence theorems for an ordinary differential system of the first order such as

$$\frac{dx_1}{X_1(x_1, \cdots, x_n)} = \cdots = \frac{dx_n}{X_n(x_1, \cdots, x_n)} = \frac{dt}{1} \tag{1}$$

failed to apply in the neighborhood of the singular points $(x_1, \cdots, x_n)$ at which the functions X_i vanish simultaneously. Briot and Bouquet had examined such singular points in special cases by direct processes; from their work it appeared that integrals of the form $\psi e^{-\lambda t} = \text{const.}$ played an important role; here ψ is an ordinary convergent power series in $x_1, \cdots, x_n$ without constant terms, provided that the singular point in question is taken at the origin of coordinates. But such a function satisfies the associated linear partial differential equation

$$X_1 \frac{\partial \psi}{\partial x_1} + \cdots + X_n \frac{\partial \psi}{\partial x_n} - \lambda \psi = 0. \tag{2}$$

Poincaré undertook to make this linear partial differential equation the basis of a theory of the solutions of the system (1) near such a singular point.

The outcome is elaborated in his *Thèse* of 1879, *Sur les propriétés des fonctions définies par les équations aux différences partielles*, which almost immediately became classic. A characteristic result is the following: If by a suitable linear transformation of the variables $x_1, \cdots, x_n$ we can reduce the functions X_i to the form $\lambda_i x_i +$ terms of higher degree in $x_1, \cdots, x_n$, with $\lambda_i \neq \lambda_j$ for $i \neq j$ (general case), and if there is a line of the complex λ-plane such that the points λ_i fall on one side of it, while the origin $\lambda = 0$ falls on the other, then the formal series are convergent, and the general integral of (1) near the singular point is furnished by the n equations $\psi_i = ce^{\lambda_i t}$, $(i = 1, \cdots, n)$.

* I reviewed the earlier volume also; see *The work of Poincaré on automorphic functions*, this Bulletin, vol. 26, pp. 161–172.