THE CONDITION FOR A PFAFFIAN SYSTEM IN INVOLUTION†

BY J. M. THOMAS

1. *Introduction.* If a pfaffian system is to be employed for solving a system of partial differential equations, an integral variety on which a given set of variables are independent is desired. A pfaffian system having a *non-singular* integral variety of this type is said to be *in involution* with respect to the variables in question. A necessary and sufficient condition for a linear pfaffian system in involution has been stated by Cartan in terms of what he calls the prolonged system.‡ The present paper gives to the condition an alternative form which is obtained directly from the original system. The condition is here derived for generalized (that is, non-linear) systems.§ The paper ends (§3) with a few remarks about singular integral varieties.

2. *The Condition.* The basis of the following treatment is a theorem, which is an immediate consequence of known results and for which a simple, direct proof can also be given. It may be regarded as the basic theorem in the theory of linear, homogeneous equations. It is: *A system of linear, homogeneous equations has a solution in which a specified set of unknowns can be given arbitrary values if and only if the rank of its matrix is unaltered by the omission of the columns corresponding to those unknowns.* For the application of the theorem, we note that having a solution corresponding to arbitrary values of the given unknowns is equivalent to having a solution when each of the following sets of values is assigned to those unknowns:

$$(1)\quad (1, 0, \cdots, 0),\quad (0, 1, \cdots, 0),\quad \cdots,\quad (0, 0, \cdots, 1).$$

Let the generalized pfaffian system be

$$(2)\qquad \omega^{\lambda} = 0, \qquad (\lambda = 1, 2, \cdots, \rho),$$

† Presented to the Society under a different title, December 1, 1933.

‡ E. Cartan, Annales de l'École Normale, (3), vol. 21 (1904), pp. 153–175.

§ J. M. Thomas, *An existence theorem for generalized pfaffian systems*, this Bulletin, vol. 40, pp. 309–315. This paper will be cited as G. The reader is assumed to be familiar with its contents.