A REMARK ON THE PRECEDING NOTE BY BOCHNER

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In the preceding paper in this Bulletin,* S. Bochner has proved the following theorem: *If* $\phi(t)$ *is continuous for* $-\infty < t < \infty$ *and has the property that the inequality*

$$\left| \sum_{r=1}^{n} c_r \phi(t_r) \right| \leqq K \cdot \sup_{-\infty<\xi<\infty} \left| \sum_{r=1}^{n} c_r e^{it_r\xi} \right| \tag{1}$$

holds for any n *and for any complex-valued constants* c_r *and real constants* t_r, *then*

$$\phi(t) = \int_{-\infty}^{\infty} e^{it\xi} d\eta(\xi), \quad \textit{with} \quad V_{-\infty}^{+\infty}(\eta) \leqq K. \tag{2}$$

Here is a simple proof of the following modification of the above theorem: *If* $\phi(t)$ *is measurable and the inequality*

$$\left| \int_{-\infty}^{\infty} \phi(t) q(t) dt \right| \leqq K \cdot \max_{\xi} \left| \int_{-\infty}^{\infty} e^{it\xi} q(t) dt \right| \tag{3}$$

holds for every $q(t) \subset L$, *then there is a function of bounded variation* $\eta(\xi)$ *such that* (2) *holds almost everywhere.*†

For let A be the space of functions $q(t) \subset L$, with

$$\|q\| = \max_{\xi} \left| \int_{-\infty}^{\infty} e^{i\xi t} q(t) dt \right|,$$

and let B be the space of functions

$$\psi(t) = \int_{-\infty}^{\infty} e^{it\xi} d\eta(\xi),$$

with $\|\psi\| = V_{-\infty}^{+\infty}(\eta)$. The space A is isometric with the space A' of functions

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* Vol. 40 (1934), pp. 271–276.

† Compare with the note by F. Riesz, *Über Sätze von Stone und Bochner*, Acta Szeged, vol. 6 (1933), pp. 184–198, which suggested to me the present remark.