NOTE ON RELATIONS CONNECTING CERTAIN CASES OF CONVERGENCE IN THE MEAN*

BY DUNHAM JACKSON

1. *Introduction.* A lemma from which various inferences can be drawn with regard to the convergence of sequences of trigonometric sums reads as follows.†

LEMMA A. *If $f(x)$ is a continuous function of period 2π, $T_n(x)$ a trigonometric sum of the nth order, and*

$$G_{ns} = \int_{-\pi}^{\pi} |f(x) - T_n(x)|^s dx,$$

and if there exists a trigonometric sum $t_n(x)$ of the nth order such that $|f(x) - t_n(x)| \leqq \epsilon_n$ everywhere, then

$$|f(x) - T_n(x)| \leqq 4(nG_{ns})^{1/s} + 5\epsilon_n$$

for all values of x.

The exponent s may be any positive constant. In view of the continuity of $f(x)$, it is possible to construct approximating sums $t_n(x)$ for successive values of n and to assign corresponding upper bounds ϵ_n for the error of the approximation so that $\lim_{n\to\infty}\epsilon_n = 0$. It follows as an immediate corollary of the lemma that if a sequence of sums $T_n(x)$ has the property that $\lim_{n\to\infty} nG_{ns} = 0$, for fixed s, then $T_n(x)$ converges uniformly toward $f(x)$ as n becomes infinite. This may be regarded as constituting a relationship between the convergence properties of two measures of the discrepancy between $f(x)$ and the sum $T_n(x)$, regarded as an approximation to $f(x)$: the mean value $G_{ns}/(2\pi)$ of the sth power of the error, and the maximum value attained by the error at any single point. The latter will converge to zero if the former approaches zero *with sufficient rapidity.* On the other hand, if the maximum error approaches zero, the mean will necessarily approach zero, without further restriction.

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† See D. Jackson, *Certain problems of closest approximation*, this Bulletin vol. 39 (1933), pp. 889–908, Lemma 5.