LINEAR ASSOCIATIVE ALGEBRAS OF INFINITE ORDER WHOSE ELEMENTS SATISFY FINITE ALGEBRAIC EQUATIONS*

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1. *Introduction.* It is the purpose of this paper to investigate linear associative algebras of infinite order, whose elements satisfy finite algebraic equations with coefficients in a field Ξ. The definition† of an algebra, and the first three postulates‡ assumed, will be the same as those employed by L. E. Dickson for a finite algebra, but in place of Dickson's postulates for a finite basis we shall employ postulates IV and V as follows.

POSTULATE IV. *There exists in A a set of elements§ U of such a nature that for every $a \neq 0$ there is determined uniquely a positive integer n, a set of distinct elements $u_1, u_2, \cdots, u_n$ of U, and n non-zero elements $\xi_1, \xi_2, \cdots, \xi_n$ of Ξ such that $a = \sum_{i=1}^{n} \xi_i u_i$.*

POSTULATE V. *For every element a of A there exists a polynomial function $h(\lambda)$, with coefficients in Ξ, such that $h(a) = 0$.*

2. *Idempotent Elements; Semi-Nilpotent Algebras.*

THEOREM 1. *Every algebra A contains an idempotent element unless all its elements are nilpotent.*

For if a is any non-zero non-nilpotent element of A whose minimum|| equation is $g(\lambda) = 0$, of degree n, then the finite subalgebra $B = (a, a^2, a^3, \cdots, a^n)$ of A, contains an idempotent element.

THEOREM 2. *If an algebra A is not semi-nilpotent,¶ but con-*

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† L. E. Dickson, *Algebras and their Arithmetics*, University of Chicago Press, pp. 9–11, cited hereafter as Dickson.

‡ For the convenience of the reader, references will be made to Dickson, wherever possible, whether or not this constitutes the original source.

§ The set U is not assumed enumerable except in the example of §2, the subalgebra M of A in Theorem 11, and in Theorem 13.

|| The equation $g(\lambda) = 0$ of lowest degree, with rational coefficients and leading coefficient unity, for which $g(a) = 0$, will be called the minimum equation of a.

¶ An algebra A will be called semi-nilpotent if all its elements are nilpotent, and semi-simple if it contains no properly nilpotent elements.