DERIVATIVES, DIFFERENCE QUOTIENTS, AND TAYLOR'S FORMULA*

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1. *Introduction.* Let $f(x)$ be defined in the closed interval I. If $f(x)$ has a continuous mth derivative, it can be expanded in a Taylor's formula with $m+1$ terms plus remainder; the mth difference quotient of $f(x)$ approaches $d^m f(x)/dx^m$ uniformly. If $f(x)$ is a polynomial of degree at most $m-1$, then the mth difference quotient is identically zero. The object of the present note is to prove converses of these theorems. The results hold also in an open interval, as they hold in every closed subinterval.

2. *Difference Quotients.* Given a function $f(x)$ defined in I, the pth difference quotient is defined by the equations $\Delta_h^0 f(x) = f(x)$, and

$$\begin{aligned} \Delta_h^p f(x) &= \frac{1}{h^p}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i} f(x+ih) \\ &= \frac{1}{h}[\Delta_h^{p-1}f(x+h) - \Delta_h^{p-1}f(x)] \end{aligned} \tag{1}$$

for $p>0$. We say $\Delta_h^p f(x) \to f_p(x)$ uniformly in I if for every $\epsilon>0$ there is a $\delta>0$ such that $|\Delta_h^p f(x) - f_p(x)| < \epsilon$ for every x in I and every h, $|h|<\delta$.‡

(a) Suppose $f_0(x), f_1(x), \cdots, f_m(x)$ are defined in I, and

$$f_0(x+h) = f_0(x) + \frac{1}{1!}f_1(x)h + \cdots + \frac{1}{m!}f_m(x)h^m + R(x,h), \tag{2}$$

where $R(x,h)/h^m \to 0$ uniformly in I as $h\to 0$. If we form the pth difference quotient, we find§

* Presented to the Society, June 23, 1933.

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‡ We always suppose that the values of x under consideration (here, $x+ih$ for $i=0,\cdots,p$) lie in I.

§ If we solve the linear equations (setting $0^0=1$)

$$0^j u_0 + 1^j u_1 + \cdots + p^j u_p = p!\delta_{jp}, \qquad (j=0,\cdots,p),$$

we find $u_i = (-1)^{p-i}\binom{p}{i}$; hence $\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i} i^j = 0$, $(j<p)$, and $=p!$, $(j=p)$.