NOTE ON THE ORTHOGONALITY OF TCHEBYCHEFF POLYNOMIALS ON CONFOCAL ELLIPSES*

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In the study of polynomial expansions of analytic functions in the complex plane, two different definitions of orthogonality are current:

$$(1) \qquad \int_C n(z)p_j(z)p_k(z)\,dz = 0, \text{ or } \int_C p_j(z)q_k(z)dz = 0, \quad (j \neq k),$$

$$(2) \qquad \int_C n(z)p_j(z)\overline{p_k(z)}\,|\,dz\,| = 0, \qquad (j \neq k).$$

Definition (1) in one form or the other (the second form of (1) may be called biorthogonality) is of frequent use, for instance in connection with the Legendre polynomials,† and has the great advantage that if the functions involved are analytic, the contour of integration C may be deformed without altering the orthogonality property. Definition (2) is of importance—indeed inevitable—when one wishes to study approximation on C in the sense of least squares, and it is entirely with definition (2) that we shall be concerned in the present note. More explicitly, condition (2) may be described as orthogonality with respect to the norm function $n(z)$, ordinarily chosen as continuous and positive or at least non-negative on C.

An illustration of (1), where C is the unit circle $|z| = 1$, is the set of functions $1, z, z^2, \cdots, n(z) \equiv 1$. An illustration of (2), where C is the unit circle, is the set of functions $\cdots, z^{-2}, z^{-1}, 1, z, z^2, \cdots, n(z) \equiv 1$:

$$\int_C z^j \bar{z}^k \,|\,dz\,| = \int_C \frac{z^j}{z^k}\frac{dz}{iz} = 0, \qquad (j \neq k).$$

The connection of orthogonality in the sense of (2) with ap-

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† The reader may refer to Heine, *Kugelfunktionen*, 1878; Darboux, Journal de Mathématiques, (3), vol. 4 (1878), pp. 5–56, and pp. 377–416; Geronimus, Transactions of this Society, vol. 33 (1931), pp. 322–328.