SIMILAR SEQUENCES

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1. *Similarity*. The interesting reciprocity between the Eulerian numbers E and the pseudo-Eulerian numbers E' of Pascal,* which was noticed by Sinigallia,† is not a peculiarity of the particular sequences E, E', but is a simple instance of a general property of *all* sequences of numbers or of polynomials. According to this property, defined below and called similarity, *any* sequence is similar to an infinity of other sequences; there is no special reason for singling E' out of all the sequences similar to E.

Let $S \equiv S_n$, $S' \equiv S_n'$, $(n=0, 1, \cdots)$, be any sequences such that

$$S_n = f_n(S_0', S_1', \cdots), \qquad S_n' = f_n(S_0, S_1, \cdots),$$

for all integers $n \geqq 0$, so that the nth element S_n of S is the same function f_n of elements of S' that S_n' is of elements of S. We shall say that S, S' are *similar*, and write $S \sim S'$, and hence also $S' \sim S$.

Let $S \sim S'$, and let the relation, in symbolic or umbral notation, which enables us to express the elements of either of S, S' as functions of the elements of the other, be $R(S, S')=0$. If $R(S, S')$ is bilinear in the elements of S, S', we shall say that S, S' are *bilinearly similar*. It will be shown in §2 that all the S' bilinearly similar to any given S constitute a three-parameter family of sequences.

As an example, we state the difference equation which defines *all* E' bilinearly similar to Euler's E, using for E the notation of Lucas.‡

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* E. Pascal, Rendiconti del R. Istituto Lombardo, (2), vol. 40 (1907), pp. 461–475.

† L. Sinigallia, Rendiconti di Palermo, vol. 24 (1907), pp. 222–228.

‡ E. Lucas, *Théorie des Nombres*, Chap. 14. Sinigallia's E_{2n} is Lucas' $(-1)^n E_{2n}$. It is a great convenience in using the symbolic method to fill any gaps that may occur in a given sequence with zeros, so that the index ranges over all integers $n \geqq 0$, adding a supplementary definition to give the positions of the interpolated zeros. Thus Sinigallia's E_{2n} and Pascal's E_{2n}', $(n=0, 1, \cdots)$, would be replaced by E_n, E_n', $(n=0, 1, \cdots)$, with the supplementary defini-