ON A COVARIANT DIFFERENTIATION PROCESS: PAPER II*

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1. *Introduction*. It is the purpose of this note to construct analogs of the parameters gradient, divergence, and curl, and to establish a few of their more salient properties.

2. *Notation*. In addition to the notation used in I,† we shall employ the symbols $|$, θ to indicate ordinary covariant and Synge-Taylor differentiation, respectively.

3. *The Invariants*. Evidently, if $S(x, x', \cdots, x^{(m)})$ is a scalar, then the quantities $S_{,\alpha}$ are the components of a vector. Likewise, if $V^\alpha(x, x', \cdots, x^{(m)})$, V_α $(V_\alpha = f_{\alpha\beta}V^\beta)$ are the contravariant and covariant descriptions of a vector, and $A_{\alpha\beta}$ a second order tensor, then $V^\alpha{}_{,\alpha}$ is a scalar, $V_{\alpha,\beta} - V_{\beta,\alpha}$ a skew symmetric tensor, and $A_{\alpha\beta,\gamma} + A_{\gamma\alpha,\beta} + A_{\beta\gamma,\alpha}$ a tensor of the third order. Furthermore, if n, the dimensionality of the space, is three, and $\epsilon^{\alpha\beta\gamma}$ represents the product of $|f_{\alpha\beta}|^{-1/2}$ and the corresponding component of the contravariant e system, then $\epsilon^{\alpha\beta\gamma}\ V_{\beta,\gamma}$ is a vector. The symbols $e^{\alpha\beta\gamma}$ are skew symmetric in each pair of indices and e^{123} is unity.

A certain regularity appears if $m > 2$ or if the affine connection is that of Riemannian geometry, for example, $V_{\alpha,\beta} = f_{\alpha\gamma}V^\gamma{}_{,\beta}$, and whenever either of these cases prevails we shall employ a special symbolism. Specifically, GS shall represent the vector $S_{,\alpha}$, DV the scalar $V^\alpha{}_{,\alpha}$, and if n is three, CV the vector $\epsilon^{\alpha\beta\gamma}\ V_{\beta,\gamma}$.

In virtue of these definitions and the formal equivalence of certain of the rules of operation of our process and those of partial differentiation, we may take over many of the identities of vector analysis; for example,

$$CGS \equiv 0; \qquad DCV \equiv 0; \qquad G(S + s) \equiv GS + Gs;$$

$$D(SV) \equiv (GS)\cdot V + SDV; C(SV) \equiv (GS) \times V + SCV; \text{etc.}$$

The first of these relationships suggests the following theorem.

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† The preceding note, this Bulletin, vol. 37 (1931), p. 731.