AN INTEGRAL EQUATION WITH SYMMETRIC KERNELS*

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It is the purpose of this note to investigate conditions necessary and sufficient for the solution of the integral equation

$$(1)\qquad \int_a^b A(x, s)X(s, y)ds + \int_a^b X(x, s)B(s, y)ds = C(x, y),$$

where the kernels $A(x, y)$ and $B(x, y)$ are considered to be symmetric. We further restrict our functions of two variables to be continuous throughout the fundamental interval (a, b).

An equation of the type (1) will not in general admit a solution. However, under certain quite restrictive conditions on the function $C(x, y)$, a solution may be obtained. To determine these conditions, we may readily verify from the classical theory of integral equations that every function $C(x, y)$, for which a function $X(x, y)$ exists such that (1) is true, is developable in a uniformly convergent series

$$(2)\qquad C(x, y) = \sum_{i=1}^{\infty} \left\{ \frac{\alpha_i(x)\bar{\alpha}_i(y)}{\alpha_i} + \frac{\bar{\beta}_i(x)\beta_i(y)}{\beta_i} \right\},$$

where

$$(3)\qquad \bar{\alpha}_i(y) = \int_a^b \alpha_i(s)X(s, y)ds, \quad \bar{\beta}_i(x) = \int_a^b X(x, s)\beta_i(s)ds,$$

and where $\{\alpha_i, \alpha_i(s)\}$ and $\{\beta_i, \beta_i(s)\}$ are the characteristic values and characteristic functions of the kernels $A(x, y)$ and $B(x, y)$, respectively. To justify this conclusion, it is sufficient to note that the series for the iterated kernel

$$A^{(2)}(x, y) = \sum_{i=1}^{\infty} \frac{\alpha_i(x)\alpha_i(y)}{\alpha_i^2}, \qquad A^{(2)}(x, x) = \sum_{i=1}^{\infty} \frac{\alpha_i^2(x)}{\alpha_i^2},$$

converge uniformly and absolutely, which, in view of the boundedness of $\sum_{i=1}^{\infty} \bar{\alpha}_i^2(y)$, implies the uniform and absolute

* Presented to the Society, March 18, 1933.