RUTHERFORD ON MODULAR INVARIANTS

Modular Invariants. By D. E. Rutherford. Cambridge Mathematical Tracts, Number 27. Cambridge, University Press, and New York, Macmillan, 1932. viii+84 pp.

This little tract brings together into small compass the principal results in the theory of modular invariants (both formal and otherwise) up to 1930, thus assembling under one cover both the results of what might be called the American school—Dickson, Glenn, Sanderson, Hazlett, and others—and also the work of E. Noether based on the abstract theory of ideals, as it appears in the research of Steinitz, Artin, and van der Waerden.

The subject had its rise in 1903 in a paper by Hurwitz on the solution of higher congruences, but lay dormant until rediscovered, in another connection, by Dickson in 1907. During the next seven years, the latter developed and finished the theory of modular invariants (here called residual covariants), based on the theory of classes, developed a theory of invariants of the general linear group defined over the Galois field, $GF[p^n]$, proved the finiteness theorem for modular covariants, and made the beginnings of a theory of formal modular covariants (here called formal covariants). In 1913 appeared that short but stimulating and suggestive paper by Miss Sanderson, giving her theorem that given a formal modular invariant, i, of a system of forms under a modular group, G, defined over $GF[p^n]$, we can construct a formal modular invariant, I, such that I x is congruent to i in the field for all sets of values of coefficients in the field. In the Madison Colloquium Lectures (1914), Dickson gave a series of lectures on the theory to date. During the next eight years appeared many papers by American writers on the subject, giving treatments of special cases and proving various theorems that are more or less analogous to theorems in the classic theory of algebraic invariants. At the end of Miss Sanderson's paper, she expressed some of the formal invariants and covariants of the binary quadratic for $GF[p^n=3]$ in a symbolic form, and this small but suggestive beginning was now the source of inspiration of Miss Hazlett's paper (1921–22) on the symbolic theory of formal modular covariants of a binary form. This proved that a suitable positive, integral power of every formal modular invariant is congruent in the field to an algebraic invariant of $f(a; x)$ and certain related forms, $f(a^{p^n}; x)$, $f(a^{p^{2n}}; x)$, $\cdots$, $f(a; x^{p^n})$, $\cdots$. The same paper also proved the finiteness theorem for formal modular covariants of a system of binary forms. Then, in 1926, appeared a brief but important paper by E. Noether in which she proved the finiteness theorem for a system of n-ary forms, by using the theory of a ring of polynomials in any number of variables.

Rutherford takes all this theory—at least, all of any importance—and welds the various results and processes into a whole, putting the work of the American school into Part I (51 pages) and following this, in Part II (31 pages) by E. Noether's theorem together with as much of the theory of fields, both algebraic and transcendental, as is necessary for her proof.

Throughout the whole tract, Rutherford is very clear-cut in precisely those places where it is necessary. At the very beginning (§1), he introduces two