ON NATURAL FAMILIES OF CURVES

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1. *Introduction.* A *natural family* of curves in a Riemannian space V_N of N dimensions is a family which consists of the ∞^{2N-2} extremals of an integral $\int \mu ds$, where μ is a function of position and ds is the element of length. If a system of ∞^M curves is given in V_N, where $M \leqq 2N-2$, there does not in general exist a natural family to which all these curves belong; and the present paper is concerned with the problem of finding conditions for the existence of such a family. This problem is equivalent, as will be seen, to that of finding conditions for the possibility of representing V_N conformally on a second Riemannian space so that the curves of the second space that correspond to the curves of the given system are geodesics.

In the case where $M=2N-2$ and $N>2$, a condition is given by the extensions to Riemannian space of the so-called* theorem of Thompson and Tait and the converse theorem of Kasner. According to these theorems, a system of ∞^{2N-2} curves in V_N ($N>2$) is a natural family if and only if the ∞^{N-1} curves of the system that cut an arbitrary V_{N-1} normally are the orthogonal trajectories of a family of a single infinity of V_{N-1}'s, or in other words, form a normal congruence.† This result is applicable only when the given system contains ∞^{2N-2} curves; and it is not valid for $N=2$, since the condition is satisfied by an arbitrary system of ∞^2 curves on a surface. The result that we shall obtain in what follows is not subject to these limitations, and it yields immediately the theorem of Thompson and Tait in the cases where this is applicable.

Our argument will be mainly synthetic in form; and we shall make no attempt at a rigorous discussion of the minimum assumptions under which our results hold, contenting ourselves with supposing that the functions that we introduce, explicitly

* J. A. Schouten (Nieuw Archief, (2), vol. 15 (1928), p. 97) points out that this theorem was first given by Lipschitz.

† See E. Kasner, Transactions of this Society, vol. 11 (1910), p. 121, J. A. Schouten, loc. cit., and W. Blaschke, Nieuw Archief, (2), vol. 15 (1928), p. 202. The methods of the present paper are similar to those used by Blaschke.