NOTE ON THE LOCATION OF THE CRITICAL POINTS OF GREEN'S FUNCTION*

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1. *Introduction.* The Green's function for a multiply connected plane region enters into many investigations, so that the nature of the equipotential curves, and in particular of their singularities, is of some importance. The writer's immediate interest in the subject, for instance, arises from the study of approximation of analytic functions by polynomials; the degree of best approximation and regions of convergence of sequences of polynomials of best approximation can be described directly and simply in terms of the equipotential curves of a certain Green's function.† For an arbitrary plane region the singular points of the equipotential curves are precisely the critical points of the Green's function, that is, the points where both first partial derivatives of Green's function vanish.

The results on the location of the critical points of the Green's function which we here prove are analogous to and proved by the use of certain well known results on the location of the roots of the derivative of a polynomial, and are not difficult to establish. However, they seem not to be known even among those versed in potential theory, and are therefore set forth here.

2. *The Green's Function.* Let R be a region of the plane, and let $P:(a, b)$ be a point of R. The Green's function $G(x, y)$ for R with pole at P is the function which is harmonic‡ interior to R except at P, continuous in the corresponding closed region except at P, zero on the boundary of R, and in some neighborhood of P can be expressed as $U(x, y) - \frac{1}{2}\log\left[(x-a)^2+(y-b)^2\right]$, where $U(x, y)$ is harmonic at P. The Green's function need not exist for an arbitrary region, but, if existent, is unique. The Green's function for a region is invariant (except for a constant factor) under conformal transformation. In particular this invariant property

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† Walsh and Russell, forthcoming in the Transactions of this Society.

‡ A function is *harmonic at a point* if in some neighborhood of that point it is continuous together with its first and second partial derivatives and satisfies Laplace's equation. A function is *harmonic in a region* if it is harmonic at every point of that region. A *region* is an open connected point set.