ON THE THEORY OF FOURIER TRANSFORMS

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1. *Introduction.* Let $g(s) \subset L_2$ over $(-\infty, \infty)$. Let

$$G(u; a) = (2\pi)^{-1/2} \int_{-a}^{a} e^{-ius} g(s) ds.$$

According to the classical result of the Plancherel theory of Fourier transforms, $G(u; a)$ tends in the mean of order 2 to a function $G(u) \subset L_2$ as $a \to \infty$. This function is designated as the Fourier transform (in L_2) of $g(s)$. We shall write

$$(1) \qquad T\{u; g(s)\} = G(u) = \operatorname*{l.i.m}_{a\to\infty} G(u; a).$$

The functions $g(s)$ and $G(u)$ are reciprocal in the sense that

$$(2) \qquad g(s) = T\{s; G(-u)\},$$

which means that

$$(3) \quad g(s) = \operatorname*{l.i.m}_{a\to\infty} g(s; a); \quad g(s; a) = (2\pi)^{-1/2} \int_{-a}^{a} e^{ius} G(u) du.$$

As an immediate consequence of the convergence in the mean of $G(u; a)$ and $g(s; a)$, we have, almost everywhere,

$$(4) \qquad G(u) = (2\pi)^{-1/2} \frac{d}{du} \int_{-\infty}^{\infty} g(s) \frac{1 - e^{-ist}}{is} ds,$$

$$(5) \qquad g(s) = (2\pi)^{-1/2} \frac{d}{ds} \int_{-\infty}^{\infty} G(u) \frac{1 - e^{ius}}{-iu} du.$$

The reciprocity between g and G is expressed here in terms not involving the convergence in the mean.

Assume now that $g(s) \subset L_p$, $1 < p < 2$. Denote by p' the *conjugate* exponent, $p' = p/(p-1)$, $1/p + 1/p' = 1$. Titchmarsh* showed that Plancherel's theory can be extended, at least in part, to the present case. Indeed he proved that $G(u; a)$ con-

* E. C. Titchmarsh, *A contribution to the theory of Fourier transforms*, Proceedings of the London Mathematical Society, (2), vol. 23 (1925), pp. 279–289. We have slightly modified Titchmarsh's notation inasmuch as he deals with cosine- and sine-transforms, while we use exponential transforms.