FURTHER MEAN-VALUE THEOREMS*

BY MORRIS MARDEN

The present note is a sequel to my recent article† in which certain mean-value theorems due to Weierstrass and Fekete were generalized. The generalizations resulted from replacing a positive real weight-function by one assuming only values in the angular region $0 \leqq \arg w \leqq \gamma < \pi$. Here the generalizations will be extended in such a manner as to yield analogous theorems in which the weight-function takes on arbitrary real values (Corollary 2, Theorem 3) or more generally any values in the double angle $0 \leqq \arg (\pm w) \leqq \gamma < \pi$ (Corollary 1, Theorem 3). Incidentally, these extensions yield (as Corollary 3, Theorem 3) the generalization of the Gauss-Lucas theorem which formed the principal result of another paper.‡

In what follows we shall denote by $f(Z)$ the point set $w = f(z)$ obtained on letting point z vary over the point set Z; by $\Delta \arg (Z - p)$ the magnitude of the smallest angle, with vertex at the point p, enclosing the point set Z; by $K(Z)$ the smallest convex region containing set Z, and, finally, by $S(Z, \theta)$ the star-shaped region composed of all points from which the set Z subtends an angle of not less than θ. The regions $K(Z)$ and $S(Z, \theta)$ can also be defined as the loci of all points p which satisfy respectively the inequalities $\Delta \arg (Z-p) \geqq \pi$, $\Delta \arg (Z-p) \geqq \theta$. Obviously, $S(Z, \theta) \equiv S(K(Z), \theta)$ and hence $S(Z, \theta)$ always contains $K(Z)$. Finally, in what follows, the two rectifiable curves

$$C: \quad z = z(s), \quad a \leqq s \leqq b; \qquad \Gamma: \quad \lambda = \lambda(t), \quad \alpha \leqq t \leqq \beta,$$

will serve as the curves of integration, and, unless further qualified, all functions introduced hereafter will be supposed to be continuous on these curves except perhaps for a finite number of finite jumps.

* Presented to the Society, April 15, 1933.

† M. Marden, this Bulletin, vol. 38 (1932), pp. 434–441.

‡ M. Marden, *On the zeros of certain rational functions*, Transactions of this Society, vol. 32 (1930), pp. 658–668.