A SELF-DUAL SEPTIMIC POSSESSING SEVEN OF EACH KIND OF THE SIMPLE SINGULARITIES, AND AUTOPOLAR BY SEVEN RECTANGULAR HYPERBOLAS AND A CIRCLE*

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The equation of the locus in homogeneous rectangular coordinates, x, y, z, is developed under the assumptions:

i. The seven cusps are distributed at equal intervals about the unit-circle, their cuspidal tangents concurrent at (0, 0, 1).

ii. The curve is invariant under rotations about the origin through angles $2\pi k/7$, $(k=1, 2, 3, \cdots, 6)$.

iii. Each cuspidal tangent is an axis of symmetry.

As an immediate consequence of ii and iii, one node and one point of inflexion must lie on each cuspidal tangent.

The most general equation of a locus of the seventh order is

$$Af_7(x, y) + Bzf_6(x, y) + Cz^2f_5(x, y) + Dz^3f_4(x, y) + Ez^4f_3(x, y) + Fz^5f_2(x, y) + Gz^6f_1(x, y) + Hz^7 = 0.$$

The various $f_i(x, y)$ are homogeneous polynomial functions of degree i in x, y, which must be individually absolutely invariant under the rotations about the origin through angles $2k\pi/7$. Now $f_i(x, y)=0$ represents i straight lines through the origin. But it is manifestly impossible for i straight lines to be invariant under rotations of $2\pi/7$ if $i<7$, unless they are the isotropic lines $(x^2+y^2)^h=0$. Therefore we may write $f_6(x, y)\equiv(x^2+y^2)^3$, $f_4(x, y)\equiv(x^2+y^2)^2$, $f_2(x, y)\equiv(x^2+y^2)$, $C=0$, $E=0$, $F=0$, since the equation is to be real and to represent a real locus. Accordingly in polar coordinates the locus is represented by

$$A\rho^7 \cos (7\theta + \alpha) + B\rho^6 + D\rho^4 + F\rho^2 + H = 0.$$

Placing a cusp at $\rho=1$, $\theta=0$, causes α to vanish.

When $\theta=0$, the roots of $A\rho^7+B\rho^6+D\rho^4+F\rho^2+H=0$ must include a triple root for $\rho=1$, the cusp, and a double root for $\rho=a$, the node. It is next shown that for just one essential set of real values for A, B, D, F, H, the equation can have such roots. Assume

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