A NOTE ON THE DICKSON THEOREM ON UNIVERSAL TERNARIES*

BY A. A. ALBERT

1. *Introduction.* A form f with integer coefficients in integer variables is called *universal* if it represents *all* positive and negative integers. Evidently, since f is homogeneous, it represents zero for the variables all zero. In case $f=0$ for integral values of the variables not all zero f is called a zero form.

L. E. Dickson† has given a number-theoretic proof of his theorem that *every universal ternary quadratic form is a zero form.* But his proof is highly technical and consequently quite long and complicated. In the present note I shall give an almost trivial rational proof of Dickson's result. I shall also prove a generalization of his theorem for ternaries over any non-modular field F.

2. *Quadratic Forms over F.* Let F be any non-modular field and let $f(x_1, \cdots, x_n)$ be an n-ary quadratic form over F. Then we shall call f a *zero form* if $f=0$ for $x_1, \cdots, x_n$ in F and not all zero. We shall also say that, if every ρ in F is represented by f for $x_1, \cdots, x_n$ in F, the form f is universal over F.

It is well known‡ that there exists a non-singular linear transformation $x_i=\sum a_{ij}X_j$ with a_{ij} in F such that

$$f(x_1, \cdots, x_n) \equiv \phi(X_1, X_2, \cdots, X_n) \equiv \sum_{i=1}^{r} g_iX_i^2 + 0\cdot \sum_{j=r+1}^{n} X_j^2,$$

with $g_i\neq 0$ in F. The integer r is the rank of f. Evidently f is a zero form if and only if ϕ is a zero form. But if $r<n$, the form ϕ vanishes for any X_n in F, if $X_1= \cdots =X_r=0$.

THEOREM 1. *Every n-ary of rank $r<n$ is a zero form. Every n-ary of rank n is equivalent to*

$$g_1X_1^2 + g_2X_2^2 + \cdots + g_nX_n^2, \qquad (g_i \text{ in } F),$$

with g_i all not zero.

---

* Presented to the Society, April 15, 1933.

† See his *Studies in the Theory of Numbers*, pp. 17–21.

‡ See Dickson, *Modern Algebraic Theories*, p. 69