# CONVERGENCE FACTORS FOR DOUBLE SERIES*

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1. *Introduction*. By a theorem due originally to Frobenius† if the power series $y(z)=\sum_{i=0}^{\infty}a_i z^i$ has the unit circle as circle of convergence, and if $\sum_{i=0}^{\infty}a_i$ is summable by Cesàro's first mean with the value $s$, then $\lim y(z)=s$ as $z\rightarrow+1$ along any path lying between two fixed chords intersecting at $z=+1$. This theorem has been considerably extended, in the field of double series notably by Bromwich and Hardy,‡ and by C. N. Moore.§ The former proved that if $f(x, y)=\sum_{i,j=0}^{\infty}a_{ij}x^i y^j$, and if $|S_{ij}^{(k)}|$, the $k$th Hölder mean of $\sum a_{ij}$, is bounded for all values of $i$ and $j$, and $\lim_{i,j\rightarrow\infty} S_{ij}^{(k)}=s$, then also $\lim_{x,y\rightarrow 1} f(x, y)=s$. More particular reference will presently be made to Moore's paper, his theorems being the starting point for the present article. Robison,‖ also, has given necessary and sufficient conditions for the regularity of a transformation applied to a double sequence.

The writer, in a paper on series of the form $y(z)=\sum_{i=0}^{\infty}a_i z^{f(i)}$, gave sufficient conditions on $f(i)$ so that $\lim_{z\rightarrow 1} y(z)=s$.¶ The present paper deals with double series of the type

$$J(z, w) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}z^{f(i)}w^{g(j)},$$

where $z$, $w$ are complex variables, and $f(i)$, $g(j)$ are logarithmico-exponential functions,** called for brevity $L$-functions. Sufficient conditions on $f(i)$, $g(j)$ will be given so that if $\sum a_{ij}$ is summable $(C, r-1)$ with the value $s$, then $J(z, w)$ will be convergent for $|z|<1$, $|w|<1$, and $\lim_{(z,w)\rightarrow(1,1)} J(z, w)=s$.

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† Journal für Mathematik, vol. 89 (1880), p. 262.

‡ Proceedings of the London Mathematical Society, (2), vol. 2 (1904), p. 161.

§ Transactions of this Society, vol. 29 (1927), p. 227.

‖ Transactions of this Society, vol. 28 (1926), p. 50.

¶ American Journal of Mathematics, vol. 53 (1931), p. 817.

** Hardy, *Orders of Infinity*.