NOTE ON SETS OF POSITIVE MEASURE*

BY HENRY BLUMBERG

A recurring question concerning (L-measurable) sets of positive measure is what properties they have in common with the linear interval. The following theorem is concerned with such a property, stated for sets of n-dimensional positive measure lying in euclidean n space.

THEOREM. *Let $A_1, A_2, \cdots, A_p$ be p sets of positive measure lying in euclidean n space. Then there exist p n-dimensional spheres $S_1, S_2, \cdots, S_p$ such that for every set of p points s_ν, $(\nu=1,2,\cdots,p)$, belonging respectively to these spheres, there exists a set of p points a_ν, $(\nu=1,2,\cdots,p)$, lying respectively in $A_1, A_2, \cdots, A_p$, such that the sets $\{a_\nu\}$ and $\{s_\nu\}$ are congruent. Moreover, there exists a set of p congruent spheres S_ν satisfying the condition just stated and a positive number δ such that for every selected $\{s_\nu\}$, with s_ν belonging to S_ν, the associated $\{a_\nu\}$ may be so chosen that a_1 ranges over a set of measure $>\delta$.*

PROOF. Since A_ν is of positive measure, there is a sphere S'_ν in which the relative measure of A_ν is greater than $1-\epsilon$, where ϵ is a given positive number less than 1; that is, $m(A_\nu, S'_\nu)/m(S'_\nu) > 1-\epsilon$, $m(A)$ standing for the measure of A. We may suppose, and we do so for simplicity of statement, that all the S'_ν, $(\nu=1,\cdots,p)$, are equal, and we denote their common measure by μ, and their respective centers by c_ν. Let ρ be a positive number such that if a sphere of measure μ is translated a distance $<\rho$, the part belonging to the sphere in both positions is of measure $>(1-\epsilon)\mu$. Denote by v_ν, $(\nu=1,\cdots,p-1)$, the vector represented by the segment $c_\nu c_{\nu+1}$; and let w_ν, $(\nu=1,\cdots,p)$, be a given set of n-dimensional vectors, each of length $<\rho$. If a set A (or point a) is given a displacement represented by the vector $\pm v$, we denote the set (or point) in its new position by $A\pm(v)$ (or $a\pm(v)$). Writing $A_\nu S'_\nu = T^{(\nu)}$ and $T'=T'_1$, we set

$$T'_1 + (v_1 - w_1 + w_2) = T'_2\,;\ T'_2 T'' = T''_1\,;$$

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