A NOTE ON THE EQUIVALENCE OF ALGEBRAS OF DEGREE TWO*

BY A. A. ALBERT

The simplest type of normal simple algebra over any non-modular field F is the cyclic algebra of degree two (order four)

(1) $Q(\alpha, \beta) = (1, i, j, ij), ji = - ij, i^2 = \alpha \neq 0, j^2 = \beta \neq 0,$

(α and β in F), the so-called generalized quaternion algebra over F. Of great importance in the theory of linear algebras is the problem of finding conditions that two given normal simple algebras of the same degree shall be equivalent. But this problem has not, as yet, been explicitly solved even for the above simplest case of algebras of degree two except when F is an algebraic field.† The purpose of this brief note is to give a simplification of my own previous results for rational algebras of degree two and thereby to give simple explicit conditions that any two generalized quaternion algebras over any non-modular field F shall be equivalent.

We consider an algebra $Q(\alpha, \beta)$. A quantity x in $Q(\alpha, \beta)$ but not in F has the property $x^2 = \gamma$ in F if and only if

(2) $x = \xi_1 i + (\xi_2 + \xi_3 i)j, \ x^2 = \gamma = \xi_1^2\alpha + \xi_2^2\beta - \xi_3^2\alpha\beta.$

Suppose first that another algebra $Q(\gamma, \delta)$ has the property $\gamma = \xi_1^2\alpha$ for ξ_1 in F. Then, as is well known,‡ we have the following lemma.

LEMMA. *If $\gamma = \xi_1^2\alpha$, then $Q(\alpha, \beta)$ is equivalent to $Q(\gamma, \delta)$ if and only if*

---

* Presented to the Society, February 25, 1933.

† See this Bulletin, vol. 36 (1930), pp. 535–540, for algebras of degree two over any algebraic number field, and Hasse's arithmetic invariant theory in the Transactions of this Society, vol. 34 (1932), for algebras of degree n over any algebraic number field. Hasse's conditions of course have no meaning for the case we are considering.

‡ The case $n = 2$ of Hasse's Theorem (2.12) (loc. cit. p. 173) if $\alpha \neq \epsilon^2$ for any ϵ of F. If $\alpha = \epsilon^2$, then both of the above algebras are total matric algebras and are equivalent for any β and δ. But also the equation $\delta = (\xi_4^2 - \xi_5^2\alpha)\beta \neq 0$ has the solution $2\xi_4 = (1 + \delta\beta^{-1})$, $2\xi_5\epsilon = (1 - \delta\beta^{-1})$.