LANE ON PROJECTIVE DIFFERENTIAL GEOMETRY

Projective Differential Geometry of Curves and Surfaces. By Ernest Preston Lane. Chicago, The University of Chicago Press, 1932. xi+321 pp.

The author's aim in writing this treatise is clearly set forth in the preface: "Projective differential geometry is largely a product of the first three decades of the twentieth century. The theory has been developed in five or more different languages, by three or four well recognized methods, in various notations, and partly in journals not readily accessible to all." It is Lane's purpose "to organize an exposition of these researches." He "desires to coordinate the results achieved on both sides of the Atlantic so as to make the work of the European geometers more readily accessible to American students, and so as to make better known to others the accomplishments of the American school."

Differential geometry is defined as "the theory of the properties of a configuration in the neighborhood of a general one of its elements." Such properties were known long before the name of projective differential geometry was invented. In fact algebraic geometry, which is as old as differential geometry, has been and is using projective differential notions very frequently.

An excellent confirmation of this statement may be found in the first chapter of the book under review, in connection with the theory of space cubics and the nullsystems attached to them, which also serves as a classical example in algebraic geometry. These ideas are extended to space curves in general and we find among others the interesting theorem: *The nullsystem of the osculating linear complex at a point P of a curve C is the same as the nullsystem of the osculating twisted cubic of C at P.*

Here is a typical proposition of projective differential geometry: *In a space S_n the $\infty^{n(n+2)}$ integral curves of a given differential equation*

$$x^{(n+1)} + (n+1)p_1x^{(n)} + \cdots + p_{n+1}x = 0$$

are all projectively equivalent. A differential equation (of this sort) defines a curve in the space S_n except for a projective transformation, and a geometric theory based on the equation must be a projective theory.

The study of projective *invariants* and *covariants* in differential geometry is, of course, of great importance. A projective invariant of an integral curve is defined as a function of the coefficients of the preceding differential equation, and of their derivatives, which is invariant under the total transformation $x=\lambda(t)\zeta$ (λ scalar $\neq 0$). Every absolutely invariant equation connecting these invariants is independent of the analytic representation of the curve and expresses a projective geometric property of the curve. A *covariant* defines a curve whose points are in (1, 1) correspondence with the points of the original curve and obtained from the latter by a projective geometric construction.

The systematic development of these theories goes back to the remarkable geometrician, the lamented Wilczynski, who in his memorable projective differential geometry published by Teubner in 1906, mentions the admirable theory developed by Halphen. In presenting these theories Wilczynski however followed his own methods, "both for the sake of uniformity and simplicity."