ERROR EXPRESSIONS FOR CERTAIN CONTINUED FRACTIONS*

BY RAYMOND GARVER

Every discussion of the subject shows that the error committed in taking the nth convergent p_n/q_n as the true value of the simple continued fraction

$$(1)\quad a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\cdots\frac{1}{a_n +}\cdots, \qquad (a_i \text{ positive integers}),$$

is less than $1/(q_n q_{n+1})$, and greater than $a_{n+2}/(q_n q_{n+2})$ in absolute value. Further, p_n/q_n is smaller than the true value if n is odd, and larger if n is even, provided the continued fraction does not terminate.

The purpose of the present paper is to supply alternate error limits for the important case when the continued fraction is the expansion of the square root of an integer N. These limits have the advantage that they do not require the computation of any convergent beyond the nth. Further, when applied as corrections to the nth convergent, they practically always lead to a much closer approximation than do the general error limits. Our results may be summarized as follows:

Given $N^{1/2}$ expressed in the form (1), with nth convergent p_n/q_n, and with k defined† as $|p_n^2 - Nq_n^2|$. If n is even, the error committed in taking p_n/q_n as the value of $N^{1/2}$ is greater than $k/(2p_n q_n)$, and less than $k/(2q_n^2 N)^{1/2}$, in absolute value. If n is odd, this statement is true with "greater than" and "less

* Presented to the Society, August 31, 1932.

† We might affix a subscript to k, but it does not seem necessary. If p_n and q_n are large the arithmetic required to compute k might be considerable if it were necessary to multiply out $p_n^2 - Nq_n^2$. This is not, however, usually necessary. For, as is well known, the values that k can take on come in cycles, and the complete cycle can be formed easily from small values of n in many cases. It is also known that k is less than $2N^{1/2}$, which will render the complete squaring of p_n and q_n unnecessary even when the complete cycle of values has not been formed. Finally, if $n = mc$, where m is a positive integer and c is the number of partial quotients in the period of the continued fraction expansion of $N^{1/2}$, then k is 1. This case will receive special mention at the end of the present paper.