SOLUTION OF THE ZARANKIEWICZ PROBLEM*

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1. *Introduction.* In 1925 C. Zarankiewicz† proposed the following problem: *Is every acyclic continuous curve‡ homeomorphic with some proper subset of itself?* It is the purpose of this paper to show that the above question is to be answered in the negative.

Our result will depend upon the following theorem.

THEOREM. *The acyclic continuous curve S is homeomorphic with no proper subset of itself if it contains a set K such that* (1) *each point of K is a fixed point with respect to any* (1, 1) *bicontinuous transformation of S into a subset of itself*; *and* (2) *each point of S of* (*Urysohn-Menger*) *order* >1 *lies on an arc of S whose end points are points of K.*

PROOF. Let p be any point of S of order >1. There is an arc a_1a_2 in S which contains p and whose end points are points of K. Let T be any (1, 1) bicontinuous transformation of S into a subset of itself. Since $T(a_1)=a_1$ and $T(a_2)=a_2$, and since there is just one arc in S from a_1 to a_2, T must carry a_1a_2 into itself. Hence there is a point q of a_1a_2 such that $T(q)=p$. Thus the subset of S into which T carries S must contain all points of S of order >1. As these points are dense in S, this subset must be S itself.

Our problem, then, is to construct an acyclic continuous curve which satisfies the conditions of the above theorem. We shall first define certain auxiliary sets $E_{x_1x_2\cdots x_k}$.

2. *Definition of the Sets* $E_{x_1x_2\cdots x_k}$. Within a linear interval ab choose points a_n so that $a_{n+1}<a_n$ and $\lim a_n=a$. Within each interval $a_{n+1}a_n$ choose points $a_{n,m}$ so that $a_{n,m}<a_{n,m+1}$ and $\lim_{m=\infty} a_{n,m}=a_n$. At each point a_n and $a_{n,m}$ erect a perpendicular to ab. Take these perpendiculars so that for any $\epsilon>0$ only a finite number of them have a length $>\epsilon$. The set of points obtained in this way will be called a set E_1. The point a will be

* Presented to the Society, October 29, 1932.

† Fundamenta Mathematicae, vol. 7, p. 381, problem 37.

‡ The term *continuous curve* is used throughout the present article to mean a compact, locally connected, metric continuum.