CESÀRO SUMMABILITY OF DOUBLE SERIES

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1. *Definitions and Notation.* The familiar Cesàro transform of a double series $\sum_{i,j=1}^{\infty} u_{ij}$ is given by

$$\tilde{S}_{mn}^{(\alpha,\beta)} = S_{mn}^{(\alpha,\beta)} \Big/ \left[\binom{m+\alpha-1}{\alpha} \binom{n+\beta-1}{\beta} \right], \tag{1}$$

where

$$S_{mn}^{(\alpha,\beta)} = \sum_{i,j=1}^{m,n} \binom{m+\alpha-i}{\alpha} \binom{n+\beta-j}{\beta} u_{ij}. \tag{2}$$

The series $\sum u_{ij}$ is said to be summable $(C; \alpha, \beta)$ to S if we have $\lim_{m,n\to\infty} \tilde{S}_{mn}^{(\alpha,\beta)} = S$; to be bounded $(C; \alpha, \beta)$ if $|\tilde{S}_{mn}^{(\alpha,\beta)}| <$ const. for all values of m and n; and to be ultimately bounded $(C; \alpha, \beta)$ if $\limsup_{m,n\to\infty} |\tilde{S}_{mn}^{(\alpha,\beta)}| < \infty$. This definition holds for all values of α and β, real or complex, except negative integers, the binomial coefficients being defined as usual in terms of the gamma function; however we shall be concerned only with real orders greater than -1.

A special but important type of double series is that for which u_{ij} is factorable, say

$$u_{ij} = v_i w_j, \qquad (i, j = 1, 2, 3, \cdots). \tag{3}$$

Defining

$$V_m^{(\alpha)} = \sum_{i=1}^{m} \binom{m+\alpha-i}{\alpha} v_i; \quad W_n^{(\beta)} = \sum_{j=1}^{n} \binom{n+\beta-j}{\beta} w_j, \tag{4}$$

we have, by (2), when (3) holds, $S_{mn}^{(\alpha,\beta)} = V_m^{(\alpha)} W_n^{(\beta)}$, so that, by (1),

$$\tilde{S}_{mn}^{(\alpha,\beta)} = \tilde{V}_m^{(\alpha)} \tilde{W}_n^{(\beta)}, \tag{5}$$

where the factors in the right member of (5) are, respectively, the (C, α) transform of $\sum v_i$ and the (C, β) transform of $\sum w_j$.

2. *Examples.* The relation (5) enables us to obtain very easily examples illustrating the following statements.