THE POISSON INTEGRAL FOR FUNCTIONS WITH POSITIVE REAL PART*

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The aim of this paper is to derive a Poisson integral representation valid for all functions $g(\lambda)$ which are regular in the right λ-half-plane, and for which†

$$\Re g(\lambda) \geqq 0$$

in that half-plane, and in particular for such functions $g(\lambda)$ which are real for real values of λ. This latter class of functions as well as their Poisson integral representations play a fundamental role in the theory of alternating current networks.‡ The resulting Poisson integral representation (equation (10)) is a very simple one and closely connected with the theory of Stieltjes' continued fractions. These facts seem to justify the elementary derivation presented here, though the equivalent Poisson integral for the unit circle is well known.

Herglotz§ has proved¶ the theorem: *Every function $f(z)$ regular in the interior of the unit circle with real part not negative (and only such functions) can be represented as*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z}\, d\mu(\alpha) + k, \tag{1}$$

the integral being taken in the Stieltjes sense, μ being a non-decreasing bounded function, k a pure imaginary constant.

Taking the real part we obtain the Poisson integral for any

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† $\Re$ means "real part of."

‡ W. Cauer, Jahresberichte der deutschen Mathematiker Vereinigung, 1929; Mathematische Annalen, vol. 105 (1931); vol. 106 (1932); O. Brune, Journal of Mathematics and Physics, Mass. Inst. of Tech., 1931.

§ G. Herglotz, *Über Potenzreihen mit positivem reellen Teil im Einheitskreis*, Leipziger Berichte, vol. 63 (1911). Herglotz also gives an explicit expression for $\mu(\phi)$ in terms of the coefficients of the power series for $f(z)$.

¶ Another proof follows by theorems of Helly, Wiener Sitzungsberichte, vol. 121 (IIa) (1912), p. 283 and 288. See also Evans, *The Logarithmic Potential*, 1927, p. 46; Bray, Annals of Mathematics, (2), vol. 20 (1919), p. 180, Theorem 3; T. H. Hildebrandt, this Bulletin, vol. 28 (1922), pp. 53–58.