NOTE ON DEFINING PROPERTIES OF HARMONIC FUNCTIONS

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1. *Introduction.* The object of this note is to extend slightly, and to give new proofs for, two theorems characterizing harmonic functions. The first of these is the classical theorem discovered independently by Bôcher and Koebe.†

THEOREM A. *If $u(x, y)$ is continuous with its partial derivatives of the first order in an open continuum D, and if for every circle C contained in D*

$$\int_C \frac{\partial u}{\partial n} ds = 0,$$

then $u(x, y)$ is harmonic in D.

The second is Gergen's recent generalization of the above theorem.‡

THEOREM B. *If $v(x, y)$ is harmonic and positive in D, if $u(x, y)$ is continuous with its partial derivatives of the first order in D, and if*

$$\int_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds = 0$$

for every circle C contained in D, then $u(x, y)$ is harmonic in D.

We shall prove the following two theorems.

THEOREM 1. *If $u(x, y)$ is continuous with its partial derivatives of the first order in D, and if for every point (x, y) in D*

$$(1) \qquad \lim_{r\to 0} \frac{1}{2\pi r}\int_0^{2\pi}\left[\frac{\partial u(x + r\cos\theta, y + r\sin\theta)}{\partial r}\right]d\theta = 0,$$

then $u(x, y)$ is harmonic in D.

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† See, for instance, Kellogg, *Foundations of Potential Theory*, 1929, p. 227.

‡ J. J. Gergen, *Note on a theorem of Bôcher and Koebe*, this Bulletin, vol. 37 (1931), pp. 591–596.